第一次作业 (Jordan Curve Theorem)

请于**二月二十八日周五（校历第二周）** 当堂上交本次作业。请独立完成。如有参与讨论者，请引用或致谢他们。

1. Let Γ be a Jordan curve in the plane, i.e. the image of the unit circle $C = \{(x, y) : x^2 + y^2 = 1\}$ under an injective continuous mapping γ into $\mathbb{R}^2$.

   (A) A Jordan curve is said to be a Jordan polygon if C can be covered by finitely many arcs on each of which γ has the form: $\gamma(\cos t, \sin t) = (\lambda t + \mu, \rho t + \sigma)$ with constants $\lambda, \mu, \rho, \sigma$. Prove that for any Jordan polygon Γ, $\mathbb{R}^2 \backslash \Gamma$ consists of two connected components. Here we use the original definition that two points are in the same connected component if and only if they can be joined by a continuous path (image of [0,1]).

   (B) Every Jordan curve Γ can be approximated arbitrarily well by a Jordan polygon Γ'.

   (C) Let Γ be a Jordan polygon. Denote by B the connected component of $\mathbb{R}^2 \backslash \Gamma$ which is bounded in $\mathbb{R}^2$. Then B contains a disc D such that the boundary of D contains two points $\gamma(a)$ and $\gamma(b)$ such that $|a - b| \geq \sqrt{3}$.

   (D) Consider a Jordan polygon Γ and two points a, b, belonging to the same component, X, of $\mathbb{R}^2 \backslash \Gamma$. For every line segment S, contained in X except for its endpoints, $X \backslash S$ consists of two components, as is immediate from (A). Let the distance between Γ and $\{a, b\}$ be at least 1 and assume that for every S of length less than 2, a and b are in the same component of $X \backslash S$. Then there is a continuous path Π from a to b such that $\mathrm{dist}(\Pi, \Gamma) \geq 1$.

   (E) Prove that for any Jordan curve Γ in the plane, $\mathbb{R}^2 \backslash \Gamma$ consists of two connected components and Γ is their common boundary. Here for any open subset $B \subset \mathbb{R}^2$, the boundary of B is defined to be $\overline{B} \backslash B$.