第四次作业 (Monotone Convergence Theorem)

请于**三月二十一日周五（校历第五周）** 当堂上交本次作业。请独立完成。如有参与讨论者，请引用或致谢他们。

1. For any open subset $\mathcal{O} \subset \mathbb{R}$, write it as an at-most countable union of disjoint open intervals I_j. We define its measure to be
$$m(\mathcal{O}) := \sum m(I_j).$$
Generally, for any set E R, we define its exterior measure as
$$m_*(E) := \inf_{\mathcal{O} \supset E} m(\mathcal{O}).$$
A set $E \in \mathbb{R}$ is Lebesgue measurable if and only if given any $\epsilon > 0$ there exists an open set $\mathcal{O} \supset E$ such that $m_*(\mathcal{O} - E) < \epsilon$. A function $f : \to [0, \infty]$ is called measurable if $\{x : f(x) < c\}$ is measurable for all $c \in \mathbb{R}$. Let g be a measurable function on $[a, b]$ such that $0 < f < M$. Look at the partition $0 = y_0 < \cdots < y_n = M$ such that $y_j = \frac{j}{n} M$, and let
$$E_j := \{x : y_{j-1} < f(x) < y_j\}.$$
Then E_j is measurable since f is measurable. We define
$$\phi_n := \sum_{j=1}^{n} y_{j-1} \chi_{E_j},$$
where $\chi_{E_j}(x) = 1$ if $x \in E_j$ and 0 if $x \notin E_j$. Define the integral of ϕ_n as
$$\int \phi_n := \sum_{j=1}^{n} y_{j-1} m(E_j).$$
Define the integral of f as
$$\int f := \lim_{n\to\infty} \phi_n.$$
Let f be any measurable function on $\mathbb{R}$ such that $f > 0$. Then we define
$$\int f = \sup \int g$$
over all bounded measurable g with $0 \le g \le f$ that is supported on a set with finite measure. Now prove that for a sequence of measurable functions $\{f_n\}$ on $\mathbb{R}$ such that $f_n \le f_{n+1}$ for any n. Then
$$\int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n$$
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