第九次作业 (Isothermal Coordinates)

请于**五月九日周五（校历第十二周）** 当堂上交本次作业。请独立完成。如有参与讨论者，请引用或致谢他们。

1. Let $dl^2 = Edp^2 + 2Fdpdq + Gdq^2$ be a metric defined in a domain in $\mathbb{R}^2$ with coordinates (p, q). We proceed to find new coordinates $u = u(p, q)$, $v = v(p, q)$ such that has the form

$$dl^2 = f(u, v)(du^2 + dv^2).$$

We decompose the quadratic form $dl^2 = Edp^2 + 2Fdpdq + Gdq^2$ into factors as follows.

$$dl^2 = \left(\sqrt{E}dp + \frac{F + i\sqrt{g}}{\sqrt{E}}dq\right)\left(\sqrt{E}dp + \frac{F - i\sqrt{E}}{\sqrt{E}}dq\right),$$

where $g = EG - F^2$. It suffices to find a suitable integrating factor $\lambda = \lambda(p, q)$ such that

$$\lambda\left(\sqrt{E}dp + \frac{F + i\sqrt{g}}{\sqrt{E}}dq\right) = du + iv, \ \ \bar{\lambda}\left(\sqrt{E}dp + \frac{F - i\sqrt{g}}{\sqrt{E}}dq\right) = du - iv.$$

   (a) Prove that $Lu = Lv = 0$, where

$$L = \frac{\partial}{\partial q}\left[\frac{F\frac{\partial}{\partial p} - E\frac{\partial}{\partial q}}{\sqrt{EG - F^2}}\right] + \frac{\partial}{\partial p}\left[\frac{F\frac{\partial}{\partial q} - G\frac{\partial}{\partial p}}{\sqrt{EG - F^2}}\right].$$

   (b) Prove that if E, F, G are real-analytic functions in p, q, $Lf = 0$ always has a non-trivial solution.