Complex Analysis Lecture Notes

Fall 2024

0 Introduction

This is the lecture note for *Complex Analysis*, 2024 fall, by Hanlong Fang. Typist: Yuetong Zhang.

This is not exactly the same as the paper version, and I (zyt) did some edits for typesetting and language. I also omitted some not very important part. Notice there are many ”trivial by calculation” thing, here are the parts when you really just need to do it algebraically. The important parts are not here. If I say ”see Stein pg xx”, it could still be on the paper version (actually most of the time) of the lecture note, but I don’t have enough time to copy all of them here.

Also, try this linked function to see the graph!

$$f(z) = \sin z$$

1 Lecture 1 (9.10) - History and Definitions of Complex Numbers

2 Lecture 2 (9.12) - Cauchy Integral

3 Lecture 3 (9.24) - Equivalent Concepts of Holomorphic Functions

Cauchy, 1814, the functions studied have the following special properties: Write

$$f(x) = P(x + \mathrm{i}y) + \mathrm{i}Q(x + \mathrm{i}y)$$

then

Formula 3.1 (Cauchy-Riemann)**.**

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$$
$$\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$$

Yet Cauchy assumes $Q = 0$ when $y = 0$, and in Euler’s eyes we can do the expansion

$$\sum_{n=0}^{\infty} a_n x^n$$

locally on $\mathbb{R}$, then all a_n are real.

Q1: What are other functions that satisfy the C-R equations?

A1: all these power series

$$\sum_{n=0}^{\infty} a_n z^n$$

where a_n are complex. If then write $f = P + \mathrm{i}Q$, then

$$P(z) = \frac{1}{2}\left(\sum a_n z^n + \sum \bar{a}_n \bar{z}^n\right)$$
$$Q(z) = \frac{1}{2\mathrm{i}}\left(\sum a_n z^n - \sum \bar{a}_n \bar{z}^n\right)$$

to verify the C-R equation, consider taking derivative on each term

$$\frac{\partial P}{\partial x} = \frac{1}{2}\left(\sum n a_n z^{n-1} + \sum n \bar{a}_n \bar{z}^{n-1}\right)$$

the else 3 are similar.

Q2: Any more?

A2: No. Consider the power series

$$f(z) = f(x + \mathrm{i}y) = \sum_{d=0}^{\infty}\sum_{n=0}^{d} a_{n,d-n} z^n \bar{z}^{n-d}$$

by solving the C-R equations:

$$0 = \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} =$$

(*full calculation later) Therefore, $a_{n,d-n} = 0$ if $d - n \geq 1$, then

$$\sum_{n=0}^{\infty} a_{n,0} z^n$$

Therefore, (a diagram)

New observation: if view z and $\bar{z}$ as independent variables, we observe that

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)(P + \mathrm{i}Q) = \frac{1}{2}\left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) + \frac{\mathrm{i}}{2}\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) = \frac{\partial}{\partial \bar{z}}\left(\sum\sum a_n z^n \bar{z}^{d-n}\right) = \frac{\partial f}{\partial \bar{z}}$$

for z similar. that is

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial \bar{z}}$$
$$\frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial z}$$

or its inverse

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \bar{z}} + \frac{\partial}{\partial \bar{z}}$$
$$\frac{\partial}{\partial y} = -\left(\frac{\partial}{\partial \bar{z}} - \frac{\partial}{\partial z}\right)\mathrm{i}$$

notice this is consistent with the differential

$$\mathrm{d}z = \mathrm{d}z + \mathrm{i}\,\mathrm{d}y$$
$$\mathrm{d}\bar{z} = \mathrm{d}x - \mathrm{i}\,\mathrm{d}y$$
$$\mathrm{d}x = \frac{1}{2}(\mathrm{d}z + \mathrm{d}\bar{z})$$
$$\mathrm{d}y = \frac{1}{2\mathrm{i}}(\mathrm{d}z - \mathrm{d}\bar{z})$$

Also, calculate

$$\begin{aligned}\mathrm{d}f &= \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y \\ &= \left(\frac{\partial}{\partial \bar{z}} + \frac{\partial}{\partial \bar{z}}\right) f \frac{1}{2}(\mathrm{d}z + \mathrm{d}\bar{z}) - \left(\frac{\partial}{\partial \bar{z}} - \frac{\partial}{\partial z}\right)\mathrm{i}f\frac{1}{2\mathrm{i}}(\mathrm{d}z - \mathrm{d}\bar{z}) \\ &= \frac{\partial f}{\partial z}\,\mathrm{d}z + \frac{\partial f}{\partial \bar{z}}\,\mathrm{d}\bar{z}\end{aligned}$$

Therefore, if

$$\frac{\partial f}{\partial \bar{z}} \equiv 0$$

then

$$\frac{\mathrm{d} f}{\mathrm{d} z} = \frac{\partial f}{\partial z}$$

which means $f'(z)$ exists.

(a big diagram)

All these definitions inherit from $\mathbb{R}^2$:

Definition 3.1 (convergence)**.**

Definition 3.2 (pointwise convergence)**.**

Definition 3.3 (uniform convergence)**.**

All these proven in *Analysis II*, but are the complex codomain version:

Theorem 3.1 ("Thm A")**.** *If $f_n : [a, b] \to \mathbb{C}$, $\sum f_n \rightrightarrows f$, each f_n is continuous, then*

1. *f is continuous*
2. *integration of f on $[a, b]$ can be done term by term*

$$\int f = \int \sum f_n = \sum \int f_n$$

Theorem 3.2 ("Thm B")**.** *If $f_n : [a, b] \to \mathbb{C}$, $\sum f_n \to f$, each f_n' is continuous and $\sum f_n'$ is uniformly convergent, then we can take derivative by term*

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \sum f_n = \sum \frac{\mathrm{d}}{\mathrm{d}x} f_n$$

Definition 3.4 (power series)**.** A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \ (a_n, z_0 \in \mathbb{C})$$

These also follows the very same proof as *Analysis II*:

Theorem 3.3. *Let*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \ (a_n, z_0 \in \mathbb{C})$$

be a given power series. $\exists! R \geq 0$ (possibly $+\infty$) such that if $|z - z_0| < R$, the series converges, and if $|z - z_0| > R$, the series diverges. Furthermore, the convergence is uniform and absolute on every closed disk in $\{|z - z_0| < R\}$. Moreover, if we use the convention that $1/0 = \infty$ and $1/\infty = 0$, then R is given by Hadamard's formula

$$\frac{1}{R} = \limsup |a_n|^{1/n}$$

the number R is called the radius of convergence of the power series, and the region $|z| < R$ is called the disk of convergence.

Corollary 3.4. *The radius of convergence of*

$$\sum_{n=0}^{\infty} n a_n z^{n-1}$$

is the same as $\sum a_n z^n$.

Proposition 3.5. *Let*

$$\sum_{n=0}^{\infty} a_n z^n$$

be a power series with radius of convergence $R > 0$. *Then it satisfies the C-R equation.*

Proof.

$$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = 0$$

$$\iff \frac{1}{2}\frac{\partial}{\partial x}\left(\sum a_n z^n + \sum \bar{a_n}\bar{z}^n\right) = \frac{1}{2\mathrm{i}}\frac{\partial}{\partial y}\left(\sum a_n z^n - \sum \bar{a_n}\bar{z}^n\right)$$

by corollary and theorem B, we have

$$\text{LHS} = \frac{1}{2}\left(\sum a_n \frac{\partial z^n}{\partial x} + \sum \bar{a_n}\frac{\partial \bar{z}^n}{\partial x}\right) = \frac{1}{2}\left(\sum n a_n z^{n-1} + \sum n\bar{a_n}\bar{z}^{n-1}\right)$$

similarly

$$\text{RHS} = \frac{1}{2\mathrm{i}}\left(\sum n a_n z^{n-1}\mathrm{i} + \sum n\bar{a_n}\bar{z}^{n-1}\mathrm{i}\right)$$

then LHS = RHS. similarly

$$\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0$$

□

Remark. $f'(z) = \sum n a_n z^{n-1}, \cdots$

Proposition 3.6. *Let f be a real valued function of two variables x, y defined on an open neighborhood U of $(x_0, y_0) \in \mathbb{R}^2$. If both partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous on U, then f is differentiable at (x_0, y_0).*

Proof. see *Analysis III* □

Definition 3.5. Let f be a complex valued function on $\Omega \subset \mathbb{C}$. We call f is complex differentiable at $z_0 = x + \mathrm{i}y$ if $\exists L \in \mathbb{C}$,

$$\lim_{z \to z_0}\left|\frac{f(z) - f(z_0)}{z - z_0} - L\right| = 0$$

Proposition 3.7. *Suppose $f = P + \mathrm{i}Q$ complex valued on $\Omega \subset \mathbb{C}$ satisfy C-R equations, and that $\partial P/\partial x$, $\partial P/\partial y$, $\partial Q/\partial x$, $\partial Q/\partial y$ are continuous on Ω, then f is complex differentiable on Ω.*

Proof. (*the full calculation is too long and trivial)
Write $z = x + \mathrm{i}y$, then we can write P, Q as linear approximations with $E_1, E_2 = o(|\Delta(x, y)|)$. guess L algebraically and calculate the limit, simplify with the C-R equation and find that the remaining term is

$$\left|\frac{E_1 + E_2}{\Delta(x, y)}\right| \to 0$$

□

Remark. C-R equation $\iff$

$$\frac{\partial f}{\partial \bar{z}} = 0$$

Jacobian of f is

$$\begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix}$$

the C-R equation is

$$\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix}$$

4 Lecture 4 (9.26)

Definition 4.1. A continuous curve in $\mathbb{C}$ is a continuous map $\Gamma : [a, b] \to \mathbb{C}$. The curve is called piecewise C^1 if we can divide $[a, b]$ into finitely many subintervals s.t. $\Gamma'(t)$ exists on each open interval and continuous on each closed interval ($\lim_{x \to a_i^+} \Gamma'(t)$ and $\lim_{x \to a_{i+1}^-} \Gamma'(t)$ both exist)

Remark. Stein required $\Gamma' \neq 0$

(also, define the algebra of the curves $-\Gamma$ and $\Gamma_1 + \Gamma_2$. The definition is pretty trivial but troublesome to write)

Definition 4.2. If f is defined on an open set $\Omega \subset \mathbb{C}$ and $\Gamma : [a, b] \to \mathbb{C}$ is a piecewise smooth curve with $\Gamma([a, b]) \subset \Omega$, then define the integral

$$\int_\Gamma f \,\mathrm{d}z = \sum \int_{a_i}^{a_{i+1}} f(\Gamma(t))\Gamma'(t) \,\mathrm{d}t$$

Proposition 4.1. *For a reparametrization of a curve*

$$\begin{array}{ccc} [a, b] & \xrightarrow{\Gamma} & \mathbb{C} \\ {\scriptstyle\alpha}\downarrow & \nearrow_{\tilde{\Gamma}} & \\ [\tilde{a}, \tilde{b}] & & \end{array}$$

where $\alpha(a) = \tilde{a}$ *and* $\alpha(b) = \tilde{b}$ *and* $\alpha' > 0$ *everywhere, then*

$$\int_\Gamma f(z) \,\mathrm{d}z = \int_{\tilde{\Gamma}} f(z) \,\mathrm{d}z$$

Proof. trivial by substitution □

Remark. Stein defined equivalence of curves here

Theorem 4.2 (Jordan curve theorem)**.**

$\varOmega$ be a domain with piecewise smooth boundary $\varGamma = \varGamma_1 + \varGamma_2 + \cdots + \varGamma_n$, such that $\varOmega$ is always on the left of $\varGamma$. X be a open neighborhood of $\overline{\varOmega}$.

Theorem 4.3 (Green's theorem)**.** *$g, h \in \mathrm{C}^1(X)$ then*

Formula 4.1.

$$\oint_{\varGamma} g \,\mathrm{d}x + h \,\mathrm{d}y = \iint_{\overline{\varOmega}} \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y} \right) \mathrm{d}x \,\mathrm{d}y$$

Proposition 4.4. *Let $f \in \mathrm{C}^1(X)$ and satisfy C-R equations. then*

Formula 4.2.

$$\oint_{\varGamma} f(z) \,\mathrm{d}z = 0$$

Proof. $G = f$ and $F = -\mathrm{i}f$, then

$$\oint G \,\mathrm{d}x + F \,\mathrm{d}y = \iint \left(\frac{\partial f}{\partial x} - \frac{\partial (-\mathrm{i}f)}{\partial y} \right) \mathrm{d}x \,\mathrm{d}y = \iint \left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y} \right) f \,\mathrm{d}x \,\mathrm{d}y = 0$$

□

Proposition 4.5 (Goursat)**.** *Suppose f is defined on X and f' exists everywhere on X, then*

$$\int_{\varGamma} f(z) \,\mathrm{d}z = 0$$

Proof. (only sketch here. this sketch I(zyt) wrote myself according to the paper version.)

1. For rectangle (or triangle or whatever, all the same) case, see Stein's book (pg 34 thm 1.1)
2. For the general case, first cut the region with line segments such that it's a union of simply connected regions. Then, we have

$$\oint_{\partial\Omega} = \sum \oint_{\partial\Omega_i}$$

therefore we just prove the simply connected version.

3. For a simply connected region, prove that it has a primitive.
4. First guess the primitive. By the simple connectedness, any two polygonal paths connecting two points can be transformed by adding and subtracting finite number of rectangles. (proof by combinatorics) Therefore, integral along these paths are equal. Pick z_0 and integrate along the polygonal path, and reach every point (we can do this, and proof is not shown in the paper version. My proof is: By path connectedness cover each point on the path with a disc in the open set, then finite subcover, then polygonal path) of X to get F, and this is well defined.
5. Verify that F actually works. For another point $z+h$ in a disk around z, extend the path in the disk with one horizontal and one vertical. Then imitate Stein pg 38.

□

5 Lecture 5 (10.8) - Cauchy integral formula and some applications

(I edited a lot here, because time is limited and I think there are simpler expressions)

If $\partial\Omega = C_1 - C_0$ (... some conditions), then

$$\oint_{C_1} f(z)\,\mathrm{d}z = \oint_{C_0} f(z)\,\mathrm{d}z$$

Now assume that $\partial\Omega = C$, take $a \in \Omega$ and cut off a disc C_r around it. Then

$$\oint_{\partial\Omega} f(z)\,\mathrm{d}z = \oint_{\partial C_r} f(z)\,\mathrm{d}z$$

where g is holomorphic on a neighborhood $U \setminus \overline{C_r}$ of $\Omega \setminus \overline{C_r}$. Take holomorphic f on U and $f(z)/(z-a)$ is holomorphic on $U \setminus \overline{C_r}$.

$$\oint_{\partial\Omega} \frac{f(z)}{z-a}\,\mathrm{d}z = \oint_{\partial C_r} \frac{f(z)}{z-a}\,\mathrm{d}z$$

then see Stein pg 40. We get:

Theorem 5.1. *Suppose f is holomorphic on X, then*

Formula 5.1 (Cauchy's integral formula)**.**

$$\frac{1}{2\pi\mathrm{i}} \oint_{\partial\Omega} \frac{f(z)}{z-a}\,\mathrm{d}z = f(a)$$

Remark. This is a "kernel" like Poisson kernel, Bergman kernel, Szegö kernel, ...

Theorem 5.2. *Suppose $\Omega = \{|z-a| < R\}$ and f has a complex derivative at each point of Ω, then*

$$f(z) = \sum_{n=0}^{+\infty} c_n (z-a)^n$$

is a series of functions convergent on Ω, where

$$c_n = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - a| = r} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, \mathrm{d}\zeta$$

where $0 < r < R, n \geq 0$.

Proof. see Stein pg 49 thm 4.4 □

Remark. c_n is independent of r. The power series is unique.

Theorem 5.3. *Suppose f is holomorphic on X, then*

Formula 5.2.

$$f^{(k)}(a) = \frac{k!}{2\pi \mathrm{i}} \oint_{\partial \Omega} \frac{f(z)}{(z-a)^{k+1}} \, \mathrm{d}z$$

Proof. first a lemma

Lemma 5.4. *Suppose f is complex valued on $D = (\alpha, \beta) \times (\Gamma, \delta) \subset \mathbb{R}^2$, $\partial f / \partial x$ is continuous on D, then* LHS *exists and*

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{y=c}^{d} f(x,y) \, \mathrm{d}y = \int_{y=c}^{d} \frac{\partial}{\partial x} f(x,y) \, \mathrm{d}y$$

Proof. *Analysis III*, omit □

Notice that

$$\frac{\partial}{\partial x} \left(\frac{f(\zeta)}{\zeta - z} \right) = \frac{f'(\zeta)}{(\zeta - z)^2}$$

is continuous jointly in $z \in U$ and $\zeta \in \partial \Omega$, By the above lemma and Cauchy integral formula, we have

$$\frac{\partial}{\partial x} f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial \Omega} \frac{\partial}{\partial x} \left(\frac{f(\zeta)}{\zeta - z} \right) \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \oint_{\partial \Omega} \frac{f'(\zeta)}{(\zeta - z)^2} \, \mathrm{d}\zeta$$

$$f'(z) = \frac{1}{2} \left(\frac{\partial}{\partial x} - \mathrm{i} \frac{\partial}{\partial y} \right) f(z) = \frac{\partial}{\partial x} f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial \Omega} \frac{f'(\zeta)}{(\zeta - z)^2} \, \mathrm{d}\zeta$$

$$\frac{\partial f}{\partial \bar{z}} \equiv 0$$

then do it repeatedly □

Proof. I(zyt): why not just use the fact that it's a power series?? □

Theorem 5.5 (Morera). *Suppose f is a continuous function in open disc D such that for any rectangular boundary Γ contained in D we have*

$$\oint_{\Gamma} f(z)\,\mathrm{d}z = 0$$

Then f has complex derivatives in D.

Proof. I(zyt): why do it again?? Isn't this basically part of the proof for Cauchy-Goursat?? □

Definition 5.1. Define the following as holomorphic functions on Ω:

- f is complex analytic
- f is complex differentiable
- f satisfies the C-R equation
- f integral $= 0$ for any triangular path

Corollary 5.6. *If $\{f_n\}$ holomorphic and $f_n \rightrightarrows f$, then f holomorphic*

Proof. For any point $a \in \Omega$ exists an open subset D_a s.t. $\overline{D_a} \subset \Omega$. Then f is continuous. For any rectangular boundary Γ

$$\oint_{\Gamma} f_n(z)\,\mathrm{d}z \to \oint_{\Gamma} f(z)\,\mathrm{d}z$$

then f is holomorphic. □

6 Lecture 6 (10.10) - Preparation for Residue Theory: Isolated Singularity

Theorem 6.1 (Laurent series expansion). *Let $0 \le R_1 < R_2 < +\infty$. Let f be a holomorphic function on an open annulus $\{R_1 < |z-a| < R_2\} = A$. Then*

Formula 6.1.

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a)^n$$

holds on A where

Formula 6.2.

$$c_n = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - a| = r} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\,\mathrm{d}\zeta$$

where $n \in \mathbb{Z}$ and $R_1 < r < R_2$.
Moreover, the convergence is absolute and uniform on $r_1 < |z-a| < r_2$ for any $R_1 < r_1 < r_2 < R_2$; c_n is independent of the choice of r.

Proof. Take any z with $R_1 < |z-a| < R_2$. Choose $R_1 < r_1 < r_2 < R_2$ with $r_1 < |z-a| < r_2$. By Cauchy's theorem

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - a| = r_2} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - a| = r_1} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

for the first integral of RHS we have

$$\frac{1}{\zeta - z} = \sum_{n=0}^{+\infty} \frac{(z-a)^n}{(\zeta - a)^{n+1}} \quad \text{for } |z-a| < r_2 = |\zeta - a|$$

and the second

$$\frac{1}{\zeta - z} = \sum_{n=0}^{+\infty} \frac{(\zeta-a)^n}{(z - a)^{n+1}} \quad \text{for } |\zeta-a| < r_2 = |z - a|$$

Therefore,

$$\begin{aligned} f(z) &= \sum_{n=0}^{+\infty} (z-a)^n \left(\frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - a| = r_2} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, \mathrm{d}\zeta \right) + \sum_{n=0}^{+\infty} \frac{1}{(z-a)^{n+1}} \left(\frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - a| = r_1} (\zeta - a)^n f(\zeta) \, \mathrm{d}\zeta \right) \\ &= \sum_{n=0}^{+\infty} (z-a)^n \left(\frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - a| = r_2} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, \mathrm{d}\zeta \right) + \sum_{m=-\infty}^{-1} (z-a)^m \left(\frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - a| = r_1} \frac{f(\zeta)}{(\zeta - a)^{m+1}} \, \mathrm{d}\zeta \right) \end{aligned}$$

Notice that

$$c_n = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - a| = r_i} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - a| = r} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, \mathrm{d}\zeta$$

by Cauchy theorem. □

Corollary 6.2 (Uniqueness of Coefficients of Laurent series)**.** *Suppose*

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a)^n$$

is convergent absolutely and uniformly on $r_1 < |z-a| < r_2$ *for any* $R_1 < r_1 < r_2 < R_2$*, then, for* $R_1 < r < R_2$ *and* $k \in \mathbb{Z}$*,*

$$c_k = \frac{1}{2\pi \mathrm{i}} \oint_{|z-a| = r} \frac{f(z)}{(z-a)^{k+1}} \, \mathrm{d}z$$

Proof. Fix r and take $R_1 < r_1 < r < r_2 < R_2$. Then,

$$\begin{aligned}
&\frac{1}{2\pi\mathfrak{i}} \oint_{|\zeta - a| = r} \frac{f(\zeta)}{(\zeta - a)^{k+1}} \, \mathrm{d}\zeta \\
&= \sum_{n=-\infty}^{+\infty} c_n \frac{1}{2\pi\mathfrak{i}} \oint_{|\zeta - a| = r} (\zeta - a)^{n-k-1} \, \mathrm{d}z \\
&= \sum_{n=-\infty}^{+\infty} c_n \frac{1}{2\pi\mathfrak{i}} \int_0^{2\pi} (a + r\mathfrak{e}^{\mathrm{i}\theta} - a)^{n-k-1} \, \mathrm{d}(a + r\mathfrak{e}^{\mathrm{i}\theta}) \\
&= \sum_{n=-\infty}^{+\infty} c_n \frac{r^{n-k}\mathfrak{i}}{2\pi\mathfrak{i}} \int_0^{2\pi} \mathfrak{e}^{\mathrm{i}(n-k)\theta} \, \mathrm{d}\theta \\
&= c_k
\end{aligned}$$

□

Definition 6.1. An **isolated singularity** of a holomorphic function $f(z)$ at $z = a$ means that $f(z)$ is holomorphic on some deleted open disk neighborhood $\{0 < |z - a| < R\}$ of $a \in \mathbb{C}$.

Definition 6.2. Let a be an isolated singularity of $f(z)$. Bu the Laurent series expansion, we have

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - a)^n$$

the part

$$f(z) = \sum_{n=-\infty}^{-1} c_n (z - a)^n$$

is called the **principal part** of the Laurent series of f.

Recall Cauchy's case: $f = \mathcal{F}/F$ where $F, \mathcal{F}$ are holomorphic functions.

Lemma 6.3. *Suppose F is holomorphic on $\{|z - a| < R\}$ and $F \not\equiv 0$. Then there exists $m \geq 0$ and g holomorphic on D s.t. $F(z) = (z - a)^m g(z)$ and $g(z) \neq 0$ in an open subset $\{|z - a| < r\}$.*

Proof. By series expansion

$$F(z) = \sum_{n=0}^{+\infty} c_n (z-a)^n$$

Let

$$m = \min\{n : c_n \neq 0\}$$

Then

$$F(z) = (z-a)^n \sum_{n=m}^{+\infty} c_n (z-a)^{n-m} = (z-a)^n \left(c_n + \sum_{n=m+1}^{+\infty} c_n (z-a)^{n-m-1} (z-a) \right)$$

Let

$$g(z) = c_n + \left(\sum_{n=m+1}^{+\infty} c_n (z-a)^{n-m-1} \right) (z-a)$$

and it's clear that g is convergent in $\{|z-a| < R\}$. Taking $r > 0$ small enough, we can derive that

$$\left(\sum_{n=m+1}^{+\infty} |c_n||z-a|^{n-m-1} \right) |z-a| < \left| \frac{c_n}{2} \right|$$

for all $|z-a| < r$. □

$$f = \frac{\mathcal{F}}{F} = \frac{\mathcal{F}}{(z-a)^m g} = \frac{\tilde{\mathcal{F}}}{(z-a)^m} = \sum_{n=-m}^{+\infty} c_n (z-a)^n$$

Definition 6.3. For isolated singularity of f,
When the principal part of f is:

- 0, **removable singularity**.
- finite number of nonzero terms, **pole**. The **order** of the pole is largest positive integer k s.t. $c_{-k} \neq 0$.
- infinite number of nonzero terms, **essential singularity**.

Proposition 6.4. *If $f(z)$ is bounded and holomorphic on some deleted neighborhood of a, f can be extended to a holomorphic function on a neighborhood of a*

Proof. It suffices to prove that $c_{-1} = c_{-2} \cdots = 0$.

$$|c_{-n}| = \left| \frac{1}{2\pi \mathrm{i}} \int_{|z-a|=r} f(z)(z-a)^{n-1} \, \mathrm{d}z \right| \leq \frac{1}{2\pi} \sup_{|z-c|=r} |f| r^{n-1} 2\pi r = M r^n \to 0$$

□

Proposition 6.5. *Let $a \in \mathbb{C}$ be an isolated singularity of $f(z)$, which is holomorphic on $\{0 < |z-a| < R\}$ for some $R > 0$. Then a is a pole of f* $\iff$

$$\lim_{z \to a} |f(z)| = \infty$$

Moreover, if a is a pole, the pole order k is the positive integer s.t.

$$\lim_{z\to a} \left|(z-a)^k f(z)\right|$$

is a positive number.

Proof. Suppose $c_n = 0$ for $n < -k$ and $c_{-k} = 0$ for some $k > 0$. Since $f(z) = g(z)/(z-a)^k$ where $g(z) := \sum_{n=0}^{+\infty} c_{n-k}(z-a)^n$ with $g(a) = c_{-k} \neq 0$, it follows that

$$\lim_{z\to a} |f(z)| = \lim_{z\to a} \frac{|g(z)|}{|z-a|^k} = \infty$$

$$\lim_{z\to a} \left|(z-a)^k f(z)\right| = |g(a)| = |c_{-k}| > 0$$

Now suppose that $\lim_{z\to a} |f(z)| = \infty$. Then there exists $0 < R_0 < R$ s.t. $|f(z)| \geq 1$ for all $0 < |z-a| < R_0$. Define $h = 1/f$ then $h(z) \leq 1$, and hence a is a removable singularity and can be expressed as a convergent power series $h(z) = \sum_{n=0}^{+\infty} d_n(z-a)^n$, since $\lim_{z\to a} |f(z)| = \infty$, $\lim_{z\to a} h(z) = 0$, then $h(z) = (z-a)^k g(z)$ where $g \neq 0$, then $1/g = \sum_{n=0}^{+\infty} e_n(z-a)^n$, then

$$f(z) = \frac{1}{h(z)} = \frac{1}{(z-a)^k g(z)} = \sum_{n=-k}^{+\infty} e_{n+k}(z-a)^n \quad \text{on } \{|z-a| < R_1\}$$

with $e_0 \neq 0$. □

Theorem 6.6 (Casorati-Weierstrass). *Let $a \in \mathbb{C}$ be an isolated singularity of f where f is holomorphic on $\{0 < |z-a| < R\}$. Then a is a essential singularity $\iff$ for all $0 < r < R$, $f(\{0 < |z-a| < r\})$ is dense in $\mathbb{C}$.*

$\Leftarrow$. If $z = a$ is a removable singularity, $\exists 0 < r < R$ st $|f(z)| \leq M$ for $0 < |z-a| < r$, therefore its image under f is not dense. If $|z-a|$ is a pole, then similarly $|f(z)| > 1$. □

$\Rightarrow$.] assume that $\exists 0 < r < R$ st the image is not dense. Suppose $\{|z-b| < \rho\} \cap f(\{0 < |z-a| < r\}) = \emptyset$ with $\rho \geq 0$. Define

$$g(z) = \frac{1}{f(z) - b}$$

on $\{0 < |z-a| < r\}$, then $|g(z)| \leq 1/\rho$ on B_r. Since $g \not\equiv 0$ and $z = a$ is a removable singularity of g,

$$g(z) = (z-a)^k h(z)$$

with some $k \geq 0$ and $h \neq 0$, then

$$f(z) = b + \frac{1}{(z-a)^k} \frac{1}{h(z)}$$

which makes $z = a$ a pole, s contradiction. □

7 Lecture 7 (10.15) - Residue Theorem

Definition 7.1 (residue). For an isolated singularity $z = a$ of f, which is defined on $\{0 < |z-a| < R\}$, we define the **residue** of f at $z = a$ denoted by $\operatorname{Res}_a f$, as:

$$\operatorname{Res}_a f = \frac{1}{2\pi \mathrm{i}} \oint_{|z-a|=r} f(z)\, \mathrm{d}z$$

for any $0 < r < R$.

Remark. By Laurent series, we can show that

$$\operatorname{Res}_a f = c_{-1}$$

Remark. If f has a simple pole at a, then

$$\operatorname{Res}_a f = \lim_{z \to a} (z-a) f(z)$$

if the pole is of degree order $k \geq 2$, then

$$\operatorname{Res}_a f \neq \lim_{z \to a} (z-a)^k f(z)$$

Instead,

$$\operatorname{Res}_a f = \frac{1}{(k-1)!} \lim_{z \to a} \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} \left((z-a)^k f(z) \right)$$

Theorem 7.1 (residue theorem)**.** *Suppose Ω is a bounded open subset of $\mathbb{C}$ with piecewise smooth boundary $\partial\Omega$ and U is an open neighborhood of $\overline{\Omega}$. Suppose $a_1, \cdots, a_p$ are distinct points in Ω and f is a holomorphic function on $U \setminus \{a_1, \cdots, a_p\}$. Then,*

Formula 7.1.

$$\oint_{\partial\Omega} f(z)\, \mathrm{d}z = 2\pi \mathrm{i} \sum_{j=1}^{p} \operatorname{Res}_{a_j} f$$

Proof. Denote discs around each a_j by D_j where $D_j \subset \Omega$. Consider $\Omega \setminus (\bigcup \overline{D_j}) = \tilde{\Omega}$. Then f is holomorphic on an open neighborhood of $\overline{\tilde{\Omega}}$. Then by Cauchy-Goursat theorem,

$$\oint_{\partial\Omega} f(z)\, \mathrm{d}z - \sum_{j=1}^{p} \oint_{\partial \overline{D_j}} f(z)\, \mathrm{d}z = 0$$

then

$$\oint_{\partial\Omega} f(z)\, \mathrm{d}z = 2\pi \mathrm{i} \sum_{j=1}^{p} \operatorname{Res}_{a_j} f$$

□

Remark. Definition is how you legalize your bias

Definition 7.2 (meromorphic function)**.** Suppose U is an open set of $\mathbb{C}$ and E is a discrete subset of U. A holomorphic function f defined on $U \setminus E$ is called **meromorphic** function on U if each $a \in E$ is a pole of f.

Remark. f has a pole at $z = a \iff$ near $z = a$, $f = \mathcal{F}/F$ where $\mathcal{F}, F$ are holomorphic near a.

Remark. For higher dimensions, $f(z_1, \cdots, z_n)$ is meromorphic if locally $f = \mathcal{F}/F$, where $\mathcal{F}, F$ are holomorphic near $(a_1, \cdots, a_n)$.

Remark. inverse question: if

$$U = \bigcup_{i=0}^{\infty} D_i$$

and on D_i, g_i/h_i is assigned and on $D_i \cap D_j$, $g_i/h_i - g_j/h_j$ is holomorphic, can we find a meromorphic function F on U st $F - g_i/h_i$ is holomorphic? (cousin problem: sheaf theory)

Remark. etymology:
holo-: ὅλος: whole, complete
morph-: μορφή: shape, form
mero-: μέρος: part

Remark. why residue?
one reason: weird countor into circles.

Recall:

$$\int_{-\infty}^{+\infty} \mathfrak{e}^{-\pi x^2} \, \mathrm{d}x = 1$$

Proposition 7.2. .

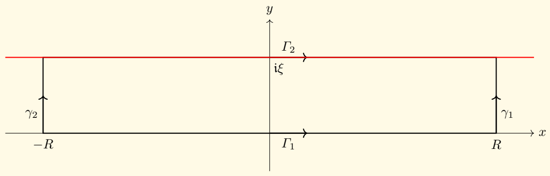
Formula 7.2.

$$\int_{-\infty}^{+\infty} \mathfrak{e}^{-\pi x^2} \mathfrak{e}^{-2\pi \mathrm{i} x \xi} \, \mathrm{d}x = \mathfrak{e}^{-\pi \xi^2}, \xi \in \mathbb{R}$$

Proof. Rewrite the formula as

$$\begin{aligned} I &= \int_{-\infty}^{+\infty} \mathfrak{e}^{-\pi x^2} \mathfrak{e}^{-2\pi \mathrm{i} x\xi} \mathfrak{e}^{\pi \xi^2} \,\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} \mathfrak{e}^{-\pi(x+\mathrm{i}\xi)^2} \,\mathrm{d}x \\ &= \int_{\mathbb{R}+\mathrm{i}\xi} \mathfrak{e}^{-\pi z^2} \,\mathrm{d}z \end{aligned}$$

consider $f(z) = e^{-\pi z^2}$ and contour $\Gamma : \Gamma_1 + \gamma_1 - \Gamma_2 - \gamma_2$.



By residue theorem,

$$\oint_{\Gamma} f(z) \,\mathrm{d}z = 0$$

then

$$\int_{\Gamma_1} f(z) \,\mathrm{d}z - \int_{\Gamma_2} f(z) \,\mathrm{d}z = \int_{\gamma_2} f(z) \,\mathrm{d}z - \int_{\gamma_1} f(z) \,\mathrm{d}z$$

where

$$\mathrm{LHS} = \int_{-R}^{R} e^{-\pi x^2} \,\mathrm{d}x - \int_{-R}^{R} e^{-\pi(x+\mathrm{i}\xi)^2} \,\mathrm{d}x$$

and for RHS,

$$\left| \int_{\gamma_j} f(z) \,\mathrm{d}z \right| \leq \mathfrak{e}^{-\pi(R^2+\xi^2)} \xi \to 0$$

then

$$\begin{aligned} &\int_{-\infty}^{+\infty} \mathfrak{e}^{-\pi^2(x+\mathrm{i}\xi)^2} \,\mathrm{d}x - \int_{-\infty}^{+\infty} \mathfrak{e}^{-\pi^2(x+\mathrm{i}\xi)^2} \,\mathrm{d}x \\ &= \lim_{R\to\infty} \left(\int_{-R}^{R} \mathfrak{e}^{-\pi^2(x+\mathrm{i}\xi)^2} \,\mathrm{d}x - \int_{-R}^{R} \mathfrak{e}^{-\pi^2 x^2} \,\mathrm{d}x \right) \\ &= \lim_{R\to\infty} \left(\int_{\gamma_2} f(z) \,\mathrm{d}z - \int_{\gamma_1} f(z) \,\mathrm{d}z \right) \\ &= 0 \end{aligned}$$

□

Integral of rational functions of sine/cosine functions over $[0, 2\pi]$

$$I := \int_0^{2\pi} R(\cos\theta, \sin\theta) \,\mathrm{d}\theta$$

method: 1. use substitution

$$z = e^{\mathrm{i}\theta} \Rightarrow \cos\theta = \frac{z + 1/z}{2}, \sin\theta = \frac{z - 1/z}{2\mathrm{i}}, \mathrm{d}\theta = \frac{\mathrm{d}z}{\mathrm{i}z}$$

then

$$I = \oint_{|z|=1} f(z) \,\mathrm{d}z$$

2. Apply residue theorem

Example 7.1.

$$a \in (0,1)$$

$$\begin{aligned} I &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2a\cos\theta + a^2} \\ &= \oint_{|z|=1} \frac{\mathrm{d}z}{\mathrm{i}z} \frac{1}{1 - 2a(z + 1/z)/2 + a^2} \\ &= \mathrm{i} \oint_{|z|=1} \mathrm{d}z \, \frac{1}{az^2 - (a^2+1)z + a} \\ &= \mathrm{i} \oint_{|z|=1} \mathrm{d}z \, \frac{1}{(az-1)(z-a)} \\ &= \frac{2\pi}{1-a^2} \end{aligned}$$

similarly, if $a > 1$, we have

$$I = \frac{2\pi}{a^2 - 1}$$

Integral of rational functions over the real line

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \, \mathrm{d}x$$

where: $\deg Q \geq \deg P + 2; Q(x) \neq 0, \forall x \in \mathbb{R}$

Theorem 7.3. *For* $\deg Q \geq \deg P + 2; Q(x) \neq 0, \forall x \in \mathbb{R}$*, we have:*

Formula 7.3.

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \, \mathrm{d}x = 2\pi\mathrm{i} \sum_{\operatorname{Im} a \geq 0} \operatorname{Res}_a \frac{P(z)}{Q(z)}$$

Proof. Let Γ_R be the contour of $[-R,R] + \{R\mathfrak{e}^{\mathrm{i}\theta}\}_{\theta=0}^{\pi} = [-R,R] + C_R$ with the usual orientation. Write

$$Q(z) = az^m \left(1 + \frac{a_1}{z} + \cdots + \frac{a_m}{z^m}\right)$$
$$P(z) = bz^n \left(1 + \frac{b_1}{z} + \cdots + \frac{b_n}{z^n}\right)$$

with $a_j, b_j \neq 0$ and $m \geq n+2$. Then

$$\begin{aligned}
&\int_{C_R} \frac{P(z)}{Q(z)}\,\mathrm{d}z \\
&= \int_{\theta=0}^{\pi} \frac{bR^n\mathfrak{e}^{\mathrm{i}n\theta}}{aR^m\mathfrak{e}^{\mathrm{i}m\theta}} \frac{1+\mathrm{o}(1)}{1+\mathrm{o}(1)} R\mathrm{i}\mathfrak{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta \\
&= \frac{1}{R^{m-n-1}} \frac{b}{a} \int_{\theta=0}^{\pi} \mathfrak{e}^{\mathrm{i}(n-m+1)\theta} \frac{1+\mathrm{o}(1)}{1+\mathrm{o}(1)}\,\mathrm{d}\theta \\
&= \mathrm{O}\left(\frac{1}{R^{m-n-1}}\right) \to 0
\end{aligned}$$

(the o(1) here is uniform and therefore the last integral is bounded by taking the smallest possible denominator and biggest possible numerator.)
Then

$$\lim_{R\to+\infty} \int_{C_R} \frac{P(z)}{Q(z)}\,\mathrm{d}z = 0$$

then

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x \\
&= \int_{\Gamma_R} \frac{P(z)}{Q(z)}\,\mathrm{d}z - \int_{C_R} \frac{P(z)}{Q(z)}\,\mathrm{d}z \\
&\to 2\pi\mathrm{i} \sum_{\mathrm{Im}\, a \geq 0} \mathrm{Res}_a \frac{P(z)}{Q(z)} + 0
\end{aligned}$$

□

Example 7.2.

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(x^2+1)^3} = 2\pi\mathrm{i}\,\mathrm{Res}_i \frac{1}{(z^2+1)^3} = 2\pi\mathrm{i}\frac{1}{2!} \frac{\mathrm{d}^2}{\mathrm{d}z^2} \frac{(z-i)^3}{(z^2+1)^3}\Big|_{z=i} = \frac{3}{8}\pi$$

8 Lecture 8 (10.22) - Integral of Sine / Cosine Times Rational Function

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \cos x\,\mathrm{d}x\,, \int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \sin x\,\mathrm{d}x$$

where $\deg Q \geq \deg P + 1$ and $Q(x) \neq 0$ except simple ones at zeros of sin or cos respectively.

Remark. I exists.

Proof. for the zeros of Q, the numerator is also zero, then they cancel
for infinity, by integration by parts

$$\int \frac{P}{Q} \operatorname{trig} \mathrm{d}x = \frac{P}{Q} \operatorname{trig} \pm \int \left(\frac{P}{Q}\right)' \operatorname{trig} \mathrm{d}x$$

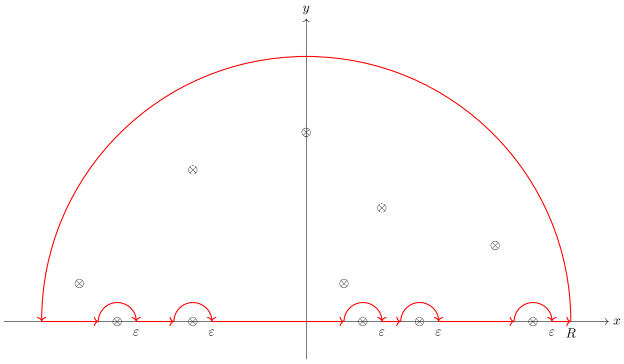
and then we have convergence of degree ≥ 2 □

we consider the function

$$f(z) = \frac{P(z)}{Q(z)} \mathfrak{e}^{\mathfrak{i}} z$$

then $\operatorname{Re} f(z) = P(x)/Q(x) \cos x, \operatorname{Im} f(z) = P(x)/Q(x) \sin x$.

now the new difficulty is: for zeros of $Q(z)$, f is ∞! therefore we consider the contour that bypass the poles: take small enough ε and big enough R st each zero of Q a_i on the real line is surrounded by a ε-semicircle, and a R-semicircle covers all the zeroes of Q on $\mathbb{C}$:



Lemma 8.1 (half residue)**.** *Let z_0 be a simple pole of a meromorphic function f. Let*

$$C_{\varepsilon,\alpha,\beta}(z_0) = \{z_0 + \varepsilon \mathfrak{e}^{\mathfrak{i}\theta} : \alpha \leq \theta \leq \beta\}$$

then

$$\lim_{\varepsilon \to 0} \int_{C_{\varepsilon,\alpha,\beta}(z_0)} f(z) \, \mathrm{d}z = (\beta - \alpha)\mathfrak{i} \operatorname{Res}_{z_0} f(z)$$

in particular, when $\alpha = 0$,$\beta = \pi$,

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi\mathfrak{i}} \int_{C_{\varepsilon,\alpha,\beta}(z_0)} f(z) \, \mathrm{d}z = \frac{1}{2} \operatorname{Res}_{z_0} f(z)$$

Proof.

$$f(z) = \frac{c_{-1}}{z - z_0} + \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_{C_{\varepsilon,\alpha,\beta}(z_0)} f(z) \,\mathrm{d}z \\
&= \lim_{\varepsilon \to 0} \int_{\theta=\alpha}^{\beta} \left(\frac{c_{-1}}{\varepsilon \mathfrak{e}^{\mathrm{i}\theta}} + \sum_{n=0}^{\infty} c_n \varepsilon^n \mathfrak{e}^{\mathrm{i}n\theta} \right) \mathrm{i}\varepsilon \mathfrak{e}^{\mathrm{i}\theta} \,\mathrm{d}\theta \\
&= \lim_{\varepsilon \to 0} \mathrm{i} \int_{\theta=\alpha}^{\beta} \left(c_{-1} + \sum_{n=0}^{\infty} c_n \varepsilon^{n+1} \mathfrak{e}^{\mathrm{i}(n+1)\theta} \right) \mathrm{d}\theta \\
&= \mathrm{i}(\beta - \alpha) c_{-1} + \lim_{\varepsilon \to 0} \mathrm{i} \sum_{n=0}^{\infty} c_n \varepsilon^{n+1} \int_{\theta=\alpha}^{\beta} \mathfrak{e}^{\mathrm{i}(n+1)\theta} \,\mathrm{d}\theta \\
&= \mathrm{i}(\beta - \alpha) c_{-1}
\end{aligned}$$

□

Lemma 8.2. *With the above setting,*

$$\lim_{R \to +\infty} \int_{C_R} \frac{P(z)}{Q(z)} \mathfrak{e}^{\mathrm{i}z} \,\mathrm{d}z = 0$$

where $C_R = \{R\mathfrak{e}^{\mathrm{i}\theta}\}_{\theta=0}^{\pi}$.

Proof. Notice that $\left|\mathfrak{e}^{\mathrm{i}z}\right| \leq 1$ when $\operatorname{Im} z \geq 0$. If $\deg Q \geq \deg P + 2$, then trivial.

$$\int_{C_R} \frac{P(z)}{Q(z)} \mathfrak{e}^{\mathrm{i}z} \,\mathrm{d}z = \left(\frac{P(z)}{Q(z)} \frac{\mathfrak{e}^{\mathrm{i}z}}{\mathrm{i}} \right)_{z=-R}^{z=R} - \int_{C_R} \left(\frac{P(z)}{Q(z)} \right)' \frac{\mathfrak{e}^{\mathrm{i}z}}{\mathrm{i}} \,\mathrm{d}z \to 0$$

□

Now we're ready for the formula:

Theorem 8.3. *For the above setting, we have:*

Formula 8.1.

$$\int_{-\infty}^{+\infty} \frac{P(z)}{Q(z)} \mathfrak{e}^{\mathrm{i}z} \,\mathrm{d}z = \pi\mathrm{i} \sum \operatorname{Res}_{a_i} \frac{P(z)}{Q(z)} \mathfrak{e}^{\mathrm{i}z} + 2\pi\mathrm{i} \sum_{\operatorname{Im} \alpha < 0} \operatorname{Res}_{\alpha} \frac{P(z)}{Q(z)} \mathfrak{e}^{\mathrm{i}z}$$

Proof. Take the above contour, Now by residue theorem, (me: I did some non rigorous notation)

$$I_{R,\varepsilon} - \pi\mathrm{i} \sum_{Q(a_i)=0} \int_{C_{a_i,\epsilon}} f(z) \,\mathrm{d}z + \int_{C_R} f(z) \,\mathrm{d}z = 2\pi\mathrm{i} \sum_{\substack{\operatorname{Im} \alpha > 0 \\ |\alpha| < R}} \operatorname{Res}_{\alpha} f$$

here $I_{R,\varepsilon}$ is the bottom line with gaps of the contour.
Then we take the limit using the lemmas, and we get the formula. □

Also, for $-\mathfrak{i}$*,*

Formula 8.2.

$$\int_{-\infty}^{+\infty} \frac{P(z)}{Q(z)} \mathfrak{e}^{-\mathfrak{i}z} \, \mathrm{d}z = \pi \mathfrak{i} \sum \mathrm{Res}_{a_i} \frac{P(z)}{Q(z)} \mathfrak{e}^{-\mathfrak{i}z} + 2\pi \mathfrak{i} \sum_{\mathrm{Im}\,\alpha<0} \mathrm{Res}_{\alpha} \frac{P(z)}{Q(z)} \mathfrak{e}^{-\mathfrak{i}z}$$

Then we take the real and imaginary part.

Example 8.1.

$$\int_{-\infty}^{+\infty} \frac{\cos x \, \mathrm{d}x}{x^2+1} = \mathrm{Re}\left(2\pi \mathfrak{i} \, \mathrm{Res}_{\mathfrak{i}} \frac{\mathfrak{e}^{\mathfrak{i}z}}{z^2+1}\right) = \frac{\pi}{e}$$

Example 8.2.

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x} \, \mathrm{d}x = \mathrm{Im}\left(\pi \mathfrak{i} \, \mathrm{Res}_0 \frac{\mathfrak{e}^{\mathfrak{i}z}}{z}\right) = \pi$$

Example 8.3.

$$\int_0^{+\infty} \frac{1-\cos x}{x^2} \, \mathrm{d}x = \frac{\pi}{2}$$

Proof. Notice

$$\int_0^{+\infty} \frac{1-\cos x}{x^2} \, \mathrm{d}x = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{1-\cos x}{x^2} \, \mathrm{d}x$$

define

$$f(z) = \frac{1-\mathfrak{e}^{\mathfrak{i}z}}{z^2}$$

and the contour is

$$[-R, -\varepsilon] - C_\varepsilon + [\varepsilon, R] + C_R$$

where C_ε and C_R are upper semicircles around 0, like how it was defined above. Then take the integral on them and use residue theorem:

$$I_{\varepsilon,R} - I_\varepsilon + I_R = 0$$

then

$$\left| \int_{C_R} f(z) \, \mathrm{d}z \right| \le \frac{2\pi}{R} \to 0$$

$$\begin{aligned} I_\varepsilon &= \int_0^\pi \frac{1-\mathfrak{e}^{\mathfrak{i}\varepsilon\mathfrak{e}^{\mathfrak{i}\theta}}}{\varepsilon \mathfrak{e}^{\mathfrak{i}\theta}} \mathfrak{i} \, \mathrm{d}\theta \\ &= \int_0^\pi \frac{1-(1+\mathfrak{i}\varepsilon\mathfrak{e}^{\mathfrak{i}\theta} + \mathrm{O}(\varepsilon^2))}{\varepsilon \mathfrak{e}^{\mathfrak{i}\theta}} \mathfrak{i} \, \mathrm{d}\theta \\ &= \int_0^\pi (1+O(\varepsilon)) \, \mathrm{d}\theta \\ &\to \pi \end{aligned}$$

(me: in the original note we made it into sin, cos. I don't think we need to do all this. Uniform is trivial anyway.) □

9 Lecture 9 (10.24) - The Use of Branches of Holomorphic Functions

Example 9.1. .

Formula 9.1.

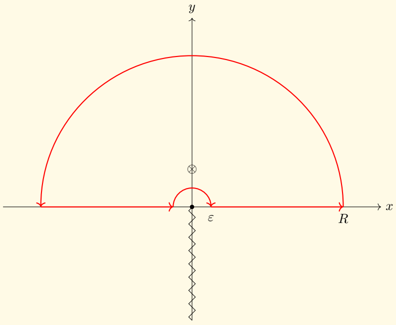
$$\int_0^{\infty} \frac{\log x}{x^2+1}\,\mathrm{d}x = 0$$

Remark. Trivially the limit exists.

Method 1. consider

$$f(z) = \frac{\log z}{z^2+1}$$

where $\log z = \log|z| + \mathfrak{i}\arg z$, $\arg z \in (-\pi/2, 3\pi/2)$, and take the indented upper half disk $C_{\epsilon,R}$:



Then by residue theorem, we have

$$\int_{+\varepsilon}^{+R} \frac{\log x}{x^2+1}\,\mathrm{d}x + \int_{\theta=0}^{\pi} \frac{\log R + \mathfrak{i}\theta}{R^2\mathfrak{e}^{2\mathfrak{i}\theta}+1}\,\mathrm{d}(R\mathfrak{e}^{\mathfrak{i}\theta}) + \int_{-R}^{-\varepsilon} \frac{\log|x| + \pi\mathfrak{i}}{x^2+1}\,\mathrm{d}x - \int_{\theta=0}^{\pi} \frac{\log\varepsilon + \mathfrak{i}\theta}{\varepsilon^2\mathfrak{e}^{2\mathfrak{i}\theta}+1}\,\mathrm{d}(\varepsilon\mathfrak{e}^{\mathfrak{i}\theta}) = \frac{1}{2\pi\mathfrak{i}}\operatorname{Res}_{\mathfrak{i}} \frac{\log z}{z^2+1}$$

Then take the limit, and

$$\int_{+\varepsilon}^{+R} \frac{\log x}{x^2+1}\,\mathrm{d}x \to \int_0^{+\infty} \frac{\log x}{x^2+1}\,\mathrm{d}x$$

$$\int_{\theta=0}^{\pi} \frac{\log R + \mathfrak{i}\theta}{R^2\mathfrak{e}^{2\mathfrak{i}\theta}+1}\,\mathrm{d}(R\mathfrak{e}^{\mathfrak{i}\theta}) = \mathrm{O}\left(\frac{\log R}{R}\right) + \mathrm{O}\left(\frac{1}{R}\right) \to 0$$

$$\int_{-R}^{-\varepsilon} \frac{\log|x| + \pi\mathfrak{i}}{x^2+1}\,\mathrm{d}x \to \int_0^{+\infty} \frac{\log x}{x^2+1}\,\mathrm{d}x + \left(\int_0^{+\infty} \frac{\log x}{x^2+1}\,\mathrm{d}x\right)\pi\mathfrak{i} = \int_0^{+\infty} \frac{\log x}{x^2+1}\,\mathrm{d}x + \frac{\pi^2}{2}\mathfrak{i}$$

$$-\int_{\theta=0}^{\pi} \frac{\log\varepsilon + \mathfrak{i}\theta}{\varepsilon^2\mathfrak{e}^{2\mathfrak{i}\theta}+1}\,\mathrm{d}(\varepsilon\mathfrak{e}^{\mathfrak{i}\theta}) = \mathrm{O}\left(\varepsilon\log\varepsilon\right) + \mathrm{O}\left(\varepsilon\right) \to 0$$

$$\frac{1}{2\pi\mathfrak{i}}\operatorname{Res}_{\mathfrak{i}} \frac{\log z}{z^2+1} = \frac{\pi^2}{2}\mathfrak{i}$$

then we have

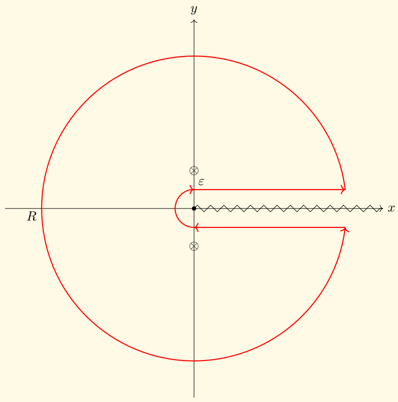
$$\int_0^{\infty} \frac{\log x}{x^2+1}\,\mathrm{d}x = 0$$

□

Method 2. we can also define this function

$$f(z) = \frac{(\log z)^2}{z^2+1}$$

where $\arg z \in (0, 2\pi)$ and this contour



□

Example 9.2. .

Formula 9.2.

$$\int_0^1 \frac{1}{x^\alpha(1-x)^{1-\alpha}} = \frac{\pi}{\sin \pi\alpha}$$

where $\alpha \in (0, 1)$

we take the function

$$f(z) = \frac{1}{z^\alpha(1-z)^{1-\alpha}}$$

where it's actually defined as

$$f(z) = \mathfrak{e}^{-\alpha \log z} \mathfrak{e}^{-(1-\alpha)\log(1-z)}$$

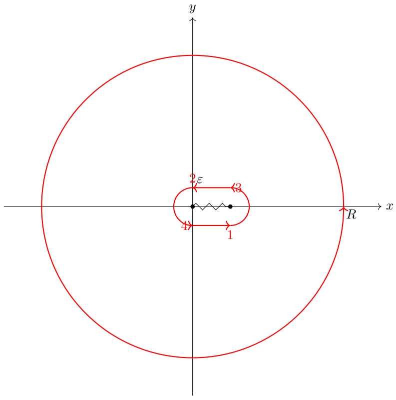
where the arg of the first log takes $(0, 2\pi)$ and the second takes $(-\pi, \pi)$.

This thing is defined only on $\mathbb{C} \setminus [0, +\infty)$, but if we naively draw a contour for this like above, the control for the right is hard.

Lemma 9.1. *$f(z)$ has a continuous extension on $\mathbb{C} \setminus [0, 1]$.*

Proof. □

Now by Morera's theorem, the extended f (we'll just call it f) is holomorphic on $\mathbb{C} \setminus [0, 1]$. Then take this contour:



By Cauchy-Goursat theorem,

$$\oint_{\Gamma_R} f(z)\,\mathrm{d}z = \oint_{\Gamma_\varepsilon} f(z)\,\mathrm{d}z$$

then

$$\begin{aligned}
\oint_{\Gamma_\varepsilon} f(z)\,\mathrm{d}z &= J_1 - J_2 + J_3 + J_4 \\
&= \int_0^1 \frac{\mathrm{d}(x - \mathrm{i}\varepsilon)}{(x-\mathrm{i}\varepsilon)^\alpha (1-(x-\mathrm{i}\varepsilon)))^{1-\alpha}} \\
&- \int_0^1 \frac{\mathrm{d}(x + \mathrm{i}\varepsilon)}{(x+\mathrm{i}\varepsilon)^\alpha (1-(x+\mathrm{i}\varepsilon)))^{1-\alpha}} \\
&+ \int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}(1+\varepsilon\mathfrak{e}^{\mathrm{i}\theta}))}{(1+\varepsilon\mathfrak{e}^{\mathrm{i}\theta})^\alpha (1-(1+\varepsilon\mathfrak{e}^{\mathrm{i}\theta})))^{1-\alpha}} \\
&+ \int_{\pi/2}^{3\pi/2} \frac{\mathrm{d}(\varepsilon\mathfrak{e}^{\mathrm{i}\theta}))}{(\varepsilon\mathfrak{e}^{\mathrm{i}\theta})^\alpha (1-(\varepsilon\mathfrak{e}^{\mathrm{i}\theta})))^{1-\alpha}}
\end{aligned}$$

Take the limit for each. Here,

$$|J_3| \le \int_{-\pi/2}^{\pi/2} \frac{\varepsilon\,\mathrm{d}\theta}{(1/2)^\alpha \varepsilon^{1-\alpha}} = \mathrm{O}((2\varepsilon)^\alpha) \to 0$$

similarly $J_4 \to 0$.

(...)

10 Lecture 10 (11.5) - Computation of Infinite Sums By Residues

11 Lecture 11 (11.7) - Special Functions

Definition 11.1 (gamma function (prototype)). .

> **Formula 11.1.**
>
> $$\Gamma(x) = \int_0^\infty t^{x-1}\mathfrak{e}^{-t}\,\mathrm{d}t$$

this formula is currently only when $x > 0$. Convergence is trivial.

Theorem 11.1 (recursion formula). *for $x > 1$, we have*

Formula 11.2.

$$\Gamma(x) = (x-1)\Gamma(x-1)$$

Proof.

$$\Gamma(x) = -t^{x-1}\mathfrak{e}^{-t}\Big|_0^\infty + (x-1)\int_0^\infty t^{x-2}\mathfrak{e}^{-t}\,\mathrm{d}t = (x-1)\Gamma(x-1)$$

□

also for integers special value,

Formula 11.3.

$$\Gamma(1) = 1, \Gamma(n) = (n-1)!$$

Lemma 11.2. *the gamma function like this*

Formula 11.4.

$$\Gamma(z) = \int_0^\infty t^{z-1}\mathfrak{e}^{-t}\,\mathrm{d}t$$

is well-defined for $z \in \mathbb{C}$ *where* $\operatorname{Re} z > 0$*, and is holomorphic.*

defined. notice that

$$t^{z-1} = \mathfrak{e}^{(\log t)(z-1)} = \mathfrak{e}^{(\log t)(\operatorname{Re} z + \mathrm{i}\operatorname{Im} z - 1)}$$

then

$$\left|t^{z-1}\right| = \left|t^{\operatorname{Re} z - 1}\right|$$

then

$$\left|\int_\varepsilon^R t^{z-1}\mathfrak{e}^{-t}\,\mathrm{d}t\right| \le \int_\varepsilon^R \left|t^{z-1}\right|\mathfrak{e}^{-t}\,\mathrm{d}t = \int_\varepsilon^R \left|t^{\operatorname{Re} z - 1}\right|\mathfrak{e}^{-t}\,\mathrm{d}t$$

and we can prove it like the $x > 0$ version. □

holomorphic. we claim that

$$\frac{\partial}{\partial x}\left(\int_0^\infty t^{z-1}\mathfrak{e}^{-t}\,\mathrm{d}t\right) = \int_0^\infty \log t \cdot t^{z-1}\mathfrak{e}^{-t}\,\mathrm{d}t$$
$$\frac{\partial}{\partial y}\left(\int_0^\infty t^{z-1}\mathfrak{e}^{-t}\,\mathrm{d}t\right) = \int_0^\infty \mathrm{i}\log t \cdot t^{z-1}\mathfrak{e}^{-t}\,\mathrm{d}t$$

it suffices to show that

$$\int_0^\infty |\log t|\left|t^{z-1}\right|\mathfrak{e}^{-t}\,\mathrm{d}t$$

exists and is continuous on $\{\operatorname{Re} z > 0\}$. Which is trivial. (me: I forgot the original theorem but the paper note just gave the proof for existence which is trivial.)
By a similar argument, we can show that the second derivatives exist and are continuous, then by C-R equation holomorphic. □

Definition 11.2 (beta function)**.** .

Formula 11.5.

$$\mathrm{B}(x,y) = \int_0^1 t^{y-1}(1-t)^{x-1}\,\mathrm{d}t$$

Lemma 11.3. .

Formula 11.6.

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

guess. Laplace transform:

$$\mathcal{L}(f)(s) = \int_0^\infty f(t)\mathfrak{e}^{-st}\,\mathrm{d}t$$

Convolution:

$$(f*g)(x) = \int_\Omega f(t)g(x-t)\,\mathrm{d}t$$

then

$$\mathrm{B}(x,y)(s) = \int_0^s t^{y-1}(s-t)^{x-1}\,\mathrm{d}t = t^{y-1} * t^{x-1}$$

by the properties of Laplacian transformation

$$\mathcal{L}(f*g) = \mathcal{L}(f)\mathcal{L}(g)$$

then

$$\mathcal{L}(\mathrm{B}(x,y)) = \mathcal{L}(t^{y-1})\mathcal{L}(t^{x-1}) = \int_0^\infty t^{y-1}\mathfrak{e}^{-st}\,\mathrm{d}t \int_0^\infty t^{x-1}\mathfrak{e}^{-st}\,\mathrm{d}t = \frac{\Gamma(y)}{s^y}\frac{\Gamma(x)}{s^x} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\mathcal{L}(t^{x+y-1})$$

then do the inverse Laplacian transform

$$\mathrm{B}(x,y)(t) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}t^{x+y-1}$$

and set $t = 1$

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

□

Proof.

$$
\begin{aligned}
\Gamma(x)\Gamma(y) &= \int_{t=0}^{\infty} t^{x-1}\mathfrak{e}^{-t}\,\mathrm{d}t \int_{u=\infty}^{\infty} u^{y-1}\mathfrak{e}^{-u}\,\mathrm{d}u \\
&= \iint t^{x-1}\mathfrak{e}^{-t}u^{y-1}\mathfrak{e}^{-u}\,\mathrm{d}t\,\mathrm{d}u \\
&\overset{u=tv}{=} \iint t^{x-1}\mathfrak{e}^{-t}(tv)^{y-1}\mathfrak{e}^{-tv}\,\mathrm{d}t\,\mathrm{d}tv \\
&= \iint t^{x-1}\mathfrak{e}^{-t}t^{y-1}v^{y-1}\mathfrak{e}^{-tv}t\,\mathrm{d}t\,\mathrm{d}v \\
&= \iint t^{x+y-1}\mathfrak{e}^{-t(1+v)}v^{y-1}\,\mathrm{d}t\,\mathrm{d}v \\
&\overset{w=t(1+v)}{=} \iint \left(\frac{w}{1+v}\right)^{x+y-1}\mathfrak{e}^{-w}v^{y-1}\,\mathrm{d}\left(\frac{w}{1+v}\right)\mathrm{d}v \\
&= \iint w^{x+y-1}\mathfrak{e}^{-w}\frac{v^{y-1}}{(1+v)^{x+y}}\,\mathrm{d}w\,\mathrm{d}v \\
&= \int_0^{\infty} w^{x+y-1}\mathfrak{e}^{-w}\,\mathrm{d}w \int_0^{\infty}\frac{v^{y-1}}{(1+v)^{x+y}}\,\mathrm{d}v \\
&= \Gamma(x+y)\mathrm{B}(x,y)
\end{aligned}
$$

□

as a special case,

Formula 11.7.

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

then we can use this formula to reflect $\Gamma(z)$ over $x = 1/2$, and extend $\Gamma(z)$ to a meromorphic function on $\mathbb{C}$.

Definition 11.3 (gamma function (full version))**.**

$$
\Gamma(z) = \begin{cases} f_1 = \displaystyle\int_0^{\infty} t^{z-1}\mathfrak{e}^{-t}\,\mathrm{d}t & \operatorname{Re} z > 0 \\ f_2 = \dfrac{\pi}{\sin \pi z}\left(\displaystyle\int_0^{\infty} t^{-z}\mathfrak{e}^{-t}\,\mathrm{d}t\right)^{-1} & \operatorname{Re} z < 1 \end{cases}
$$

Lemma 11.4. *This definition is well defined, and the result is a meromorphic function on* $\mathbb{C}$ *with simple poles at* $0, -1, -2, \cdots$

Proof. Notice $f_1 = f_2$ on $(0,1)$ and are both meromorphic, then by uniqueness of meromorphic functions, $f_1 = f_2$ on all of $\{0 < \operatorname{Re} z < 1\}$.
Consider

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

and $\Gamma(z)$ is holomorphic on $\{\operatorname{Re} z > 0\}$, then $\Gamma(z)$ has at most simple poles on $0, -1, -2, \cdots$.
Since

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

and when $\operatorname{Re} z \leq 0$, $\Gamma(1-z)$ is holomorphic and $\pi / \sin \pi z$ have simple poles on $\mathbb{Z}$, we have that $\Gamma(z)$ have simple poles on $0, -1, -2, \cdots$. □

Corollary 11.5.

$$\frac{1}{\Gamma(z)}$$

is holomorphic on $\mathbb{C}$

Example 11.1. .

Formula 11.8.

$$\Gamma(1/2) = \sqrt{\pi}$$

Proof.

$$\mathrm{B}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\pi}{\sin \pi/2} = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)}$$

□

Example 11.2. .

Formula 11.9.

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma\left(x + \frac{1}{2}\right)\Gamma(x)$$

Proof.

$$\mathrm{B}(x,x) = \int_0^1 \lambda^{x-1}(1-\lambda)^{x-1}\,\mathrm{d}\lambda = 2\int_0^{1/2} \lambda^{x-1}(1-\lambda)^{x-1}\,\mathrm{d}\lambda = 2^{1-2x}\int_0^1 (1-\mu)^{x-1}\mu^{-1/2}\,\mathrm{d}\mu = 2^{1-2x}\mathrm{B}\left(x, \frac{1}{2}\right)$$

$$\frac{\Gamma(x)\Gamma(x)}{\Gamma(2x)} = 2^{1-2x}\frac{\Gamma(x)\Gamma(1/2)}{\Gamma(x+1/2)}$$

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma\left(x + \frac{1}{2}\right)\Gamma(x)$$

□

12 Lecture 12 (11.12) - Infinite Products

Definition 12.1. (see *Analysis II*)

Theorem 12.1. *If*

$$\prod_{n=1}^{\infty}(1+z_n)$$

converges, then $z_n \to 0$.

Proof. We can assume that $z_n \neq -1$ for all n. Let

$$P_k = \prod_{n=1}^{k}(1+z_n).$$

Then $P_k \to P$, where $P \neq 0$, as $k \to +\infty$. Thus,

$$\frac{P_k}{P_{k-1}} \to 1$$

as $k \to +\infty$. □

Suppose that $|z_n| < 1$ *for all* $n = 1, 2, \ldots$ *(so that* $z_n \neq -1$*). Then*

$$\prod_{n=1}^{\infty}(1+z_n)$$

converges iff

$$\sum_{n=1}^{\infty}\log(1+z_n)$$

converges. Here, $\log$ *is the principal branch, where* $-\pi < \arg w < \pi$.

Proof. (me: This is my own proof which has nothing to do with the lecture note. The original one is pretty complicated but I think this is enough.) Define

$$S_k = \sum_{n=1}^{k}\log(1+z_n)$$

$P_k = \mathfrak{e}^{S_k}$, then S_k convergence trivially imply P_k convergence.
the problem is that $S_k = \log P_k$ only holds modulo $2\pi\mathfrak{i}$, and might not hold.
what hold is

$$\log|P_k| = \sum_{n=1}^{k}\log|1+z_n| = \operatorname{Re} S_k$$

$$\arg P_k + 2m\pi = \sum_{n=1}^{k}\arg(1+z_n) = \operatorname{Im} S_k \quad (m \in \mathbb{Z})$$

the convergence of P_k trivially implies the convergence of $\operatorname{Re} S_k$. For $\arg S_k$, consider $\arg P_k \in (\arg P - \varepsilon, \arg P + \varepsilon)$ since some k, then

$$\operatorname{Im} S_k \in (\arg P - \varepsilon, \arg P + \varepsilon) + 2m_k\pi$$

since $-\pi < \arg(1+z_n) < \pi$, when $\varepsilon < \pi/2$, m_k cannot change. Then $\operatorname{Im} S_k$ converges to some $\arg P + 2m\pi$. □

$$\prod_{n=1}^{\infty}(1+|z_n|)$$

converges iff

$$\sum_{n=1}^{\infty}|z_n|$$

converges.

Proof. See *Analysis II* □

If

$$\prod_{n=1}^{\infty}(1+|z_n|)$$

converges, then

$$\prod_{n=1}^{\infty}(1+z_n)$$

converges.

Proof.

$$\prod_{n=1}^{\infty}(1+|z_n|) \Rightarrow \sum_{n=1}^{\infty}|z_n| \Rightarrow \sum_{n=1}^{\infty}|\log(1+z_n)| \Rightarrow \sum_{n=1}^{\infty}\log(1+z_n) \Rightarrow \prod_{n=1}^{\infty}(1+z_n)$$

□

Definition 12.2. Let F_n be a sequence of functions defined on $B \subseteq \mathbb{C}$. The infinite product

$$\prod_{n=1}^{\infty} F_n(z)$$

is said to converge uniformly on B iff for some m:

1. $F_n(z) \neq 0$ for all $n \geq m$ and all $z \in B$.
2. The sequence

   $$P_k(z) := \prod_{n=m}^{k} F_n(z)$$

   converges uniformly on B to some function $P(z)$.
3. $P(z) \neq 0$ for all $z \in B$.

Lemma 12.2 (Cauchy's inequality)**.** *Suppose $f(z)$ is holomorphic on $\{|z-a|<R\}$. Then*

$$\left|f^{(n)}(a)\right| \leq \frac{n!}{r^n} \sup_{|z-a|=r} |f(z)|$$

where $0<r<R$.

Proof.

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_{|z-a|=r}\frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z$$

Thus,

$$\left|f^{(n)}(a)\right| \le \frac{n!}{2\pi}\int_{|z-a|=r}\frac{|f(z)|}{r^{n+1}}|\mathrm{d}z|$$

Using the property of integrals and bounds:

$$\left|f^{(n)}(a)\right| \le \frac{n!}{2\pi}\cdot 2\pi r\cdot\frac{\sup_{|z-a|=r}|f(z)|}{r^{n+1}} = \frac{n!}{r^n}\sup_{|z-a|=r}|f(z)|.$$

□

Remark. When $n = 0$,

$$\sup_{|z-a|<r}|f(z)| = \sup_{|z-a|=r}|f(z)|.$$

If $|f(w)| = \sup_{|z-a|=r}|f(z)|$ for $|w-a| < r$, then $f(z) = \text{constant}$.

Lemma 12.3. *Suppose that $F_n(z)$ are holomorphic functions on an open set Ω and that $\sum F_n(z)$ converges uniformly to $F(z)$ on every closed disk in Ω. Then $F(z)$ is holomorphic on Ω. Moreover, if $F_n(z) \ne 0$ for any n and $z \in \Omega$, then*

$$\frac{F'}{F} = \sum_{n=1}^{\infty}\frac{F_n'}{F_n}.$$

Proof. The holomorphicity follows by Morera's theorem.
Next, we prove that F_n' converges to F' uniformly on any closed disk in Ω. Fix a disk $\overline{D} = \{z : |z-a| \le r\}$ in Ω and take $\overline{D}_\delta = \{z : |z-a| \le r+\delta\}$ in Ω for a certain $\delta > 0$. Applying Cauchy's inequality, we can get

$$\sup_{z\in\overline{D}}|F_n' - F'| \le \frac{1}{\delta}\sup_{z\in\overline{D}_\delta}|F_n - F(z)|.$$

Then $F_n' \to F'$ uniformly on $\overline{D}$.
By definition, $F(z) \ne 0$ for all $z \in \Omega$. Define $G_m(z) := \prod_{n=1}^{m} F_n(z)$. Then $G_m(z) \to F$ uniformly on $\overline{D}$ as $m \to \infty$. Computation yields that

$$\frac{G_m'(z)}{G_m(z)} = \sum_{n=1}^{m}\frac{F_n'(z)}{F_n(z)}.$$

Thus,

$$\frac{G_m'(z)}{G_m(z)} \to \sum_{n=1}^{\infty}\frac{F_n'(z)}{F_n(z)} \to \frac{F'(z)}{F(z)} \quad \text{on } \overline{D} \text{ as } m\to\infty.$$

□

Remark. $\prod F_n \rightrightarrows F$ on $\overline{D}$ if $|F_n| < 1$ since m, and if either $\sum \log F_n$ or $\sum |F_n - 1|$ converges uniformly on $\overline{D}$.

Theorem 12.4 (Weierstrass, Hadamard, (a simple case where degree is 1)). *Let a_n be a given sequence (possibly finite) of nonzero complex numbers such that*

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^2} < \infty \tag{Δ}$$

for some $k > 0$. Then, if $g(z)$ is any entire function, the function

Formula 12.1.

$$f(z) = \mathfrak{e}^{g(z)} z^k \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \mathfrak{e}^{z/a_n}$$

is entire. In fact:

1. *The product converges uniformly on closed disks.*
2. *f has zeros at $a_1, a_2, \cdots$ and has a zero of order k at $z = 0$, but has no other zeros.*

Conversely, if f is an entire function with properties (2) and (Δ), then f can be written in the form above.

Proof. We first show that

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \mathfrak{e}^{z/a_n}$$

is entire. Since (Δ) holds, only finitely many a_n lie in $A_R = \{|z| \leq R\}$ for any fixed R.

$$\sum_{n=1}^{\infty} \left| \left(1 - \frac{z}{a_n}\right) \mathfrak{e}^{z/a_n} - 1 \right| = \sum_{|a_n| \leq R} \left| \left(1 - \frac{z}{a_n}\right) \mathfrak{e}^{z/a_n} - 1 \right| + \sum_{|a_n| > R} \left| \left(1 - \frac{z}{a_n}\right) \mathfrak{e}^{z/a_n} - 1 \right|$$

For any $|z| \leq \frac{R}{2}$, this can be bounded as

$$\leq C_1 + \sum_{n=1}^{\infty} C_2 \frac{z^2}{|a_n|^2} < \infty$$

Therefore, the product converges uniformly on $A_{R/2}$, then it converges on $\mathbb{C}$ to an entire function. □

Proof. With property 2 and Δ we can construct

$$z^k \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \mathfrak{e}^{z/a_n}$$

that satisfy exactly property 2. After division against f, what we have left is a entire function without zeros.
(me: the proof is incomplete so I think I should add the step above. This step is (kind of) in the lecture but not the note.)
Claim: Let $h(z)$ be entire with no zeros. Then there exists an entire function $g(z)$ such that $h = \mathfrak{e}^g$.
Indeed, Set $G = \frac{h'}{h}$. Integrate G to yield g. More precisely, write

$$G = a_0 + a_1 z + a_2 z^2 + \cdots$$

and let $g = a_0 z + \frac{a_1 z^2}{2} + \frac{a_2 z^3}{3} + \cdots$. Let $f = \mathfrak{e}^g$, then,

$$f' = f \cdot g' = f \cdot \frac{h'}{h}$$

then

$$\left(\frac{f}{h}\right)' = \frac{f'h - fh'}{h^2} \equiv 0$$

Thus, $h = C\mathfrak{e}^g$. □

Example 12.1. We have rigorously

Formula 12.2.

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right)$$

Proof. The zeros of $\sin z$ occur at $z = 0$ and $z = \pm n\pi$ for $n = 1, 2, \cdots$. since

$$\sum_{n=1}^{\infty} \left(\frac{1}{n\pi}\right)^2 < \infty.$$

converges, Using the Weierstrass factorization theorem, we can represent $\sin z$ as follows:

$$\begin{aligned} \sin z &= e^{g(z)} z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n\pi}\right) \mathfrak{e}^{z/n\pi} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n\pi}\right) \mathfrak{e}^{-z/n\pi} \\ &= e^{g(z)} z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) \end{aligned}$$

where $g(z)$ is an entire function.
Now solve for $g(z)$. we have,

$$(\log \sin z)' = g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}$$

and also

$$(\log \sin z)' = \cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}$$

then $g' \equiv 0$, then we get the desired result. □

Example 12.2. .

Formula 12.3.

$$\frac{1}{\Gamma(z)} = z\mathfrak{e}^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) \mathfrak{e}^{-z/n}$$

Proof. Claim: Let

$$F_n(z) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\,\mathrm{d}t = \frac{n^z n!}{z(z+1)\cdots(z+n)}$$

Proof: Let $t = ns$, so $\mathrm{d}t = n\,\mathrm{d}s$. Then:

$$\begin{aligned} F_n(z) &= n^z \int_0^1 (1-s)^n s^{z-1}\,\mathrm{d}s \\ &= n^z \left(\frac{s^z}{z}(1-s)^n \Big|_{s=0}^1 + \frac{n}{z}\int_0^1 (1-s)^{n-1} s^z\,\mathrm{d}s \right) \\ &= n^z \frac{n}{z} \int_0^1 (1-s)^{n-1} s^z\,\mathrm{d}s \\ &= \cdots \\ &= \frac{n^z n(n-1)\cdots 1}{z(z+1)\cdots(z+n)} \end{aligned}$$

Claim:

$$\lim_{n\to\infty} F_n(z) = \Gamma(z)$$

for $\operatorname{Re} z > 0$.
Proof: Consider the integral:

$$\int_n^\infty \left(1 - \frac{t}{n}\right)^n t^{z-1}\,\mathrm{d}t$$

$$0 \le \mathfrak{e}^{-t} - \left(1 - \frac{t}{n}\right)^n \le \mathfrak{e}^{-t}\left(1 - \left(1 - \frac{t^2}{n^2}\right)^n\right) \le \frac{t^2}{n}\mathfrak{e}^{-t}$$

Thus:

$$\left|\int_n^\infty \left(1 - \frac{t}{n}\right)^n t^{z-1}\,\mathrm{d}t\right| \le \int_n^\infty \mathfrak{e}^{-t}\left|t^{z-1}\right|\mathrm{d}t \to 0$$

$$\left|\int_0^n \left(\mathfrak{e}^{-t} - \left(1 - \frac{t}{n}\right)^n\right) t^{z-1}\,\mathrm{d}t\right| \le \int_0^n \frac{t^2}{n}\mathfrak{e}^{-t}\left|t^{z-1}\right|\mathrm{d}t \le \frac{1}{n}\int_0^n \mathfrak{e}^{-t}\left|t^{z+1}\right|\mathrm{d}t \to 0$$

Now, using the lemmas:

$$\begin{aligned} \frac{1}{\Gamma(z)} &= \frac{1}{\lim_{n\to\infty} F_n(z)} \\ &= z \lim_{n\to\infty} n^{-z} \prod_{k=1}^n \left(1 + \frac{z}{k}\right) \\ &= z \lim_{n\to\infty} \mathfrak{e}^{(1+1/2+\cdots+1/n-\log n)z} \prod_{k=1}^n \left(1 + \frac{z}{k}\right)\mathfrak{e}^{-z/k} \\ &= z\mathfrak{e}^{\gamma z} \prod_{k=1}^\infty \left(1 + \frac{z}{k}\right)\mathfrak{e}^{-z/k} \end{aligned}$$

□

(here down, where are these??)

Definition 12.3 (function of finite order)**.** Let f be an entire function. If there exists a positive number

ρ and constants $A, B > 0$ st

$$|f(z)| \leq A\mathfrak{e}^{B|z|^\rho}$$

then we say that f has an order of growth $\leq \rho$. We define the order of growth of f as

$$\rho_f = \inf \rho$$

Theorem 12.5. *Let f be an entire function that has an order of growth $\leq \rho$, if $z_1, z_2, \cdots$ denote the zeros of f with $z_k \neq 0$, then for all $s > \rho$ we have*

$$\sum \frac{1}{|z_k|^s} < \infty$$

Corollary 12.6. *If we know the growth of $\zeta(z)(z-1)$ is of order $\leq 1-\delta$, $0 < \delta < 1$, we have that*

$$\zeta(s) = \frac{1}{s-1}\mathfrak{e}^{a+bs}\prod_{n=1}^{\infty}\left(1+\frac{s}{2n}\right)\mathfrak{e}^{-s/2n}\prod_{\text{non-real zero } p}\left(1-\frac{s}{\rho}\right)\mathfrak{e}^{s/\rho}$$

(Jensen's formula)
Riemann zeta function
we want to extend

Formula 12.4.

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^z}$$

which is already well defined and holomorphic on $\{\operatorname{Re} z > 1\}$
Note that

$$\begin{aligned}\int_0^\infty \frac{t^{z-1}}{\mathfrak{e}^t - 1}\,\mathrm{d}t &= \int_{t=0}^{\infty}\left(\sum_{n=1}^{\infty} t^{z-1}\mathfrak{e}^{-nt}\right)\mathrm{d}t\\ &= \sum_{n=1}^{\infty}\int_{t=0}^{\infty} t^{z-1}\mathfrak{e}^{-nt}\,\mathrm{d}t\\ &= \sum_{n=1}^{\infty}\frac{1}{n^z}\int_{t=0}^{\infty} t^{z-1}\mathfrak{e}^{-t}\,\mathrm{d}t\\ &= \zeta(z)\Gamma(z)\end{aligned}$$

then we can define

Definition 12.4 (Riemann zeta function)**.** .

Formula 12.5.

$$\zeta(z) = \frac{1}{\Gamma(z)}\int_{t=0}^{\infty}\frac{t^{z-1}}{\mathfrak{e}^t - 1}\,\mathrm{d}t$$

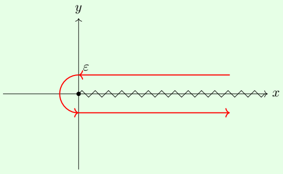
Now this is still not of any gain, since the integral touches the singularity. To find a better integral, we move the integral line away

Lemma 12.7. *Define*

Formula 12.6.

$$\hat{\zeta}(z) = \frac{\mathfrak{e}^{-\mathfrak{i}\pi z}}{2\mathfrak{i}\sin(\pi z)\Gamma(z)} \int_C \frac{\omega^{z-1}}{\mathfrak{e}^{\omega}-1}\,\mathrm{d}\omega$$

where $0 < \arg\omega < 2\pi$*. The contour is*



(convergence and meromorphicity omitted.)

Remark. The upper line approaches what we want but the lower line doesn't, because the log function has a $2\pi i$ phase difference. We just do the calculation and fix the result after the calculation.

then, $\hat{\zeta}(z)$ *coincides with* $\zeta(s)$ *on* $z = s > 1$.

Proof. (...) □

Q: What are the singularities?
A:

Proposition 12.8. $\zeta(z)$ *has a simple pole at* $z = 1$ *and is holomorphic everywhere else. The residue here is* 1.

Proof. by the form of the function

$$\zeta(z) = \Gamma(1-z)\cdots$$

the pole set is contained in $1, 2, \cdots$ and it has simple poles only. Also, $\zeta(s)$ is holomorphic for $s > 1$, then the only pole can be $z = 1$.

$$\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)} \int_{t=0}^{\infty} \frac{t^{s-1}}{\mathfrak{e}^t - 1}\,\mathrm{d}t \\
&= \frac{1}{\Gamma(s)} \left(\int_{t=0}^{1} \frac{t^{s-1}}{t} \left(1 - \frac{1}{2}t + \cdots\right) \mathrm{d}t + \int_1^{\infty} \frac{t^{s-1}}{\mathfrak{e}^t - 1}\,\mathrm{d}t \right) \\
&= \frac{1}{\Gamma(s)} \left(\frac{1}{s-1} + \mathrm{O}(1) \right) \\
&= \frac{1}{s-1} + \mathrm{O}(1)
\end{aligned}$$

as $s \to 1$. □

Q: Zeros of $\zeta(z)$
A:

Proposition 12.9. *for* $\operatorname{Re} z > 1$,

Formula 12.7.

$$\zeta(z) = \prod_p \frac{1}{1-p^{-z}}$$

Proof. $\prod_p \frac{1}{1-p^{-z}}$ converges if $\sum_p \left|\frac{1}{p^z-1}\right|$ converges. Since

$$\sum_p \left|\frac{1}{p^z-1}\right| \leq 2\sum_p \frac{1}{|p^z|} \leq 2\sum_{n=1}^{\infty} \frac{1}{n^{\operatorname{Re} z}} < \infty$$

it converges and is holomorphic for $\operatorname{Re} z > 1$. Then, it suffices to prove the equality on $s \in (1, \infty)$. In fact we have Euler's identity

$$\prod_p \frac{1}{1-p^{-s}} = \prod_p (1 + p^{-s} + p^{-2s} + \cdots) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

□

Corollary 12.10. $\zeta(z) \neq 0$ *for* $\operatorname{Re} z > 1$

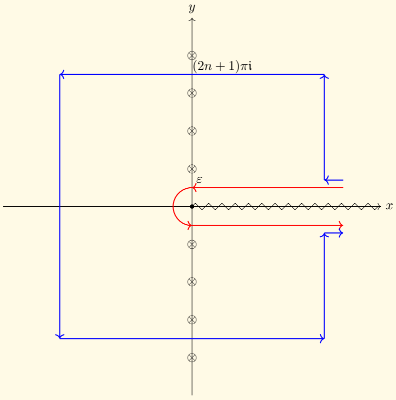
2. functional equation of $\zeta(s)$

Proposition 12.11. .

Formula 12.8.

$$\zeta(1-z) = 2^{1-z}\pi^{-z}\cos(\pi z/2)\Gamma(z)\zeta(z)$$

Proof. WLOG we can assume $z < 0$. Edit the contour before like this



$$\left(\int - \int\right) \frac{\omega^{z-1}}{e^{\omega}-1}\,\mathrm{d}\omega$$
$$= 2\pi i \sum_a \operatorname{Res}_a \frac{\omega^{z-1}}{e^{\omega}-1}$$
$$= 2\pi i \sum_{n\in\mathbb{Z}\setminus 0} \operatorname{Res}_{2\pi i n} \frac{\omega^{z-1}}{e^{\omega}-1}$$
$$= 2\pi i \sum_{n\in\mathbb{Z}\setminus 0} \lim_{\omega\to 2\pi i n} \frac{\omega^{z-1}(\omega-2\pi i n)}{e^{\omega}-1}$$
$$= 2\pi i \sum_{n\in\mathbb{Z}\setminus 0} \lim_{\omega\to 2\pi i n} \frac{\omega^{z-1}(\omega-2\pi i n)}{e^{\omega}-1}$$
$$= 2\pi i \sum_{n=1}^{+\infty} (2\pi n)^{z-1}\left(e^{(\pi i/2)(z-1)} + e^{(3\pi i/2)(z-1)}\right)$$
$$= \left(\sum_{n=1}^{+\infty} n^{z-1}\right)(2\pi)^z\left(e^{(\pi i/2)z} - e^{(3\pi i/2)z}\right)$$

also

$$\int \frac{\omega^{z-1}}{e^{\omega}-1}\,\mathrm{d}\omega = (\ldots) = O(n^{\operatorname{Re} z}) \to 0$$

then

$$\zeta(z) = -\frac{e^{-i\pi z}}{2i\sin(\pi z)\Gamma(z)}\left(\sum_{n=1}^{+\infty} n^{z-1}\right)(2\pi)^z\left(e^{(\pi i/2)z} - e^{(3\pi i/2)z}\right)$$
$$= \frac{\sin(\pi z/2)}{\sin(\pi z)}\frac{1}{\Gamma(z)}(2\pi)^z\zeta(1-z)$$
$$= 2^{z-1}\pi^z\frac{1}{\cos(\pi z/2)}\frac{1}{\Gamma(z)}\zeta(1-z)$$

□

(here down was not in the lecture but in the note.)

recall

$$\Gamma(z) = 2^{z-1}\Gamma\left(\frac{z+1}{2}\right)\Gamma\left(\frac{z}{2}\right)\pi^{-1/2}$$

then

$$\zeta(1-z)\Gamma\frac{1-z}{2}\pi^{-(1-z)/2} = \zeta(z)\Gamma\frac{z}{2}\pi^{-z/2}$$

Definition 12.5. .

Formula 12.9.

$$\xi(z) = \frac{1}{2}z(z-1)\Gamma\left(\frac{z}{2}\right)\zeta(z)\pi^{-z/2}$$

Lemma 12.12. .

Formula 12.10.

$$\xi(z) = \xi(1-z)$$

and ξ is entire.

Proof. $\Gamma(z/2)$ has simple poles at $z = 0, -2, -4, \cdots$.
Claim: $\zeta(z)$ has zeros at $z = 0, -2, -4, \cdots$.
(...) □

Proposition 12.13. *The only zeros of $\zeta(z)$ outside the strip $0 \le \operatorname{Re} z \le 1$ are at $-2, -4, \cdots$*

Proof. For $\operatorname{Re} z > 1$ $\zeta(z)$ is zero free.
Consider $\operatorname{Re} z < 0$.

$$\zeta(z) = \pi^{z-1/2}\frac{\Gamma((1-z)/2)}{\Gamma(z/2)}\zeta(1-z)$$

here $\Gamma((1-z)/2)$ is zero free, $\Gamma(z/2)$ has only poles at $-2, -4, \cdots$ □

Proposition 12.14. $\zeta(z) \neq 0$ *on* $\operatorname{Re} z = 1$

Proof. PNT □

Corollary 12.15. $\zeta(z) \neq 0$ *on* $\operatorname{Re} z = 0$

Proof. let $z = iy$
if $y \neq 0$, $\Gamma((1-iy)/2) \neq 0$, $\Gamma(iy/2)$ has no pole, $\zeta(1-iy) \neq 0$
near $y = 0$, $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1-z) \approx 1/z$, $\Gamma(z/2) \approx 1/z$, $\zeta(z) \neq 0$ □

Proposition 12.16 (Riemann's conjecture)**.** $\zeta(z) = 0$ *in* $0 \le \operatorname{Re} z \le 1$ *only if*

$$\operatorname{Re} z = \frac{1}{2}$$

13 Lecture 13 (11.19) - Conformal Mappings

Too many pictures. skipped.

14 Lecture 14 (11.21)

15 Lecture 15 (11.26)

16 Lecture 16 (12.3) - Roche

17 Lecture 17 (12.5) - Elliptic Functions

18 Lecture 18 (12.10)

19 Lecture 19 (12.17)