复变函数第二次期中考试 (a proof of prime number theorem)

1. (Product formula) Let $\Gamma(s)$ be the gamma function, and $\zeta(s)$ the Riemann zeta function.

   (a) (2 pts) Prove that for $z \in \mathbb{C}$,

   $$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \frac{1}{z-1} - \frac{1}{z} + \int_1^\infty \sum_{n=1}^\infty e^{-\pi n^2 u}\left(u^{\frac{z}{2}-1} + u^{-\frac{z}{2}-\frac{1}{2}}\right) du.$$

   (Hint: Theorem 2.2 in Page 170 of Stein; Poisson summation formula.)

   (b) (2 pts) Define

   $$\begin{aligned}\Xi(s) &:= -\frac{1}{2}\left(s^2+\frac{1}{4}\right)\pi^{-\frac{1}{4}-\frac{s}{2}\sqrt{-1}}\Gamma\left(\frac{1}{4}+\frac{s}{2}\sqrt{-1}\right)\zeta\left(\frac{1}{2}+s\sqrt{-1}\right)\\ &= \frac{1}{2} - \left(s^2+\frac{1}{4}\right)\sum_{m=0}^\infty\left(\frac{(-1)^m}{2^{2m}(2m)!}\int_1^\infty\left(\sum_{n=1}^\infty e^{-\pi n^2 u}\right)u^{-\frac{3}{4}}(\ln u)^{2m}\,du\right)s^{2m} =: \sum_{n=0}^\infty A_{2n}s^{2n}.\end{aligned}$$

   Prove that for $n = 1, 2, \cdots$,

   $$|A_{2n}| \le \frac{(\ln 2n)^{2n}}{(2n)!}.$$

   In particular, $\varliminf_{n\to\infty} \frac{-\ln|A_{2n}|}{2n\ln 2n} \ge 1$.

   (c) (1 pt) Define $M(r) := \max_{|s|\le r}|\Xi(s)|$. Prove that

   $$\varlimsup_{r\to+\infty}\frac{\ln\ln M(r)}{\ln r} \le 1.$$

   (d) (1 pt) For each $r > 0$, denote by $n(r)$ the number of the zeros of $\Xi(s)$ inside the closed disc $\{|s| \le r\}$. Prove that for any $\delta > 0$, there is a constant $C_\delta > 0$ such that

   $$n(r) \le C_\delta \cdot r^{1+\delta},\ \forall r > 0.$$

   (Hint: Apply maximal principle to $g(s) := \frac{\Xi(s)}{\left(1-\frac{s}{z_1}\right)\left(1-\frac{s}{z_2}\right)\cdots\left(1-\frac{s}{z_{n(r)}}\right)}$ where $z_1, \cdots, z_{n(r)}$ are zeros of $\Xi(s)$ inside $\{|s| \le r\}$.)

   (e) (1 pt) Denote by $\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$ the zeros of $\Xi(s)$. Prove that for any $\delta > 0$,

   $$\sum_{n=1}^\infty \frac{1}{|\alpha_n|^{1+\delta}} < \infty.$$

   (f) (1 pt) Prove that

   $$\zeta(s) = \frac{1}{s-1}e^{a+bs}\prod_{n=1}^\infty\left(1+\frac{s}{2n}\right)e^{-\frac{s}{2n}}\prod_{\rho\text{ non-real zeros of }\zeta(s)}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}$$

   (Hint: Hadamard's effective version of Weierstrass's product theorem.)

2. (1 pt) (Non-vanishing of the zeta function on the line $\Re s = 1$) Prove that the Riemann zeta function $\zeta(s)$ has no zero on the line $\Re s = 1$. (Hint: Page 185 in Stein.)

3. (Reduction to the asymptotic of $\psi_2(x)$ as $x \to +\infty$) Define arithmetic functions

$$A(x) := \sum_{\text{primes } p \le x} \ln p \cdot \ln \frac{x}{p},$$
$$\theta(x) := \sum_{\text{primes } p \le x} \ln p,$$
$$\pi(x) := \text{the number of primes less than or equal to } x.$$

(a) (1 pt) If $\theta(x) \sim x$ as $x \to +\infty$, then $\pi(x) \sim \frac{x}{\ln x}$ as $x \to +\infty$. (Hint: $\theta(x) \ge \sum_{x^{1-\epsilon} \le p \le x} \ln p$.)

(b) (1 pt) If $A(x) \sim x$ as $x \to +\infty$, then $\theta(x) \sim x$ as $x \to +\infty$. (Hint: Consider $A(x + xh) - A(x)$.)

(c) (1 pt) Prove that $\forall a > 0$,

$$\frac{1}{2\pi\sqrt{-1}} \int_{a-i\infty}^{a+i\infty} \frac{x^s}{s^2}\, ds = \begin{cases} 0 & \text{if } 0 < x < 1 \\ \ln x & \text{if } x > 1 \end{cases}.$$

(Hint: Page 192 in Stein.)

(d) (1 pt) For $a > 1$, define

$$\psi_2(x) := -\frac{1}{2\pi\sqrt{-1}} \int_{a-i\infty}^{a+i\infty} \frac{x^s}{s^2} \frac{\zeta'(s)}{\zeta(s)}\, ds.$$

Prove that $A(x) \sim \psi_2(x)$ as $x \to +\infty$. (Hint: When $\Re s > 1$, $\zeta(s) = \prod_{\text{primes } p} \frac{1}{1-p^{-s}}$)

4. (The asymptotic of $\psi_2(x)$ as $x \to +\infty$) According to Problems 1(e) and 2, $\forall \epsilon > 0$, we can choose $0 < \theta < 1$ and $\Theta > 0$ such that

$$\sum_{\text{zeros } \rho \text{ of } \zeta(s) \text{ such that } |\Re\rho| > \theta \text{ or } |\Im\rho| > \Theta} \frac{1}{|\rho|^2} < \epsilon.$$

Take $\theta < b < 1$, $a > 1$, and $1 \ll x \ll u$. Let $A = a - u\sqrt{-1}$, $B = a + u\sqrt{-1}$, $G = -u + u\sqrt{-1}$, $E = -x + x\sqrt{-1}$, $C = b + x\sqrt{-1}$, $D = b - x\sqrt{-1}$, $F = -x - x\sqrt{-1}$, $H = -u - u\sqrt{-1}$. Consider the contour integral

$$-\frac{1}{2\pi\sqrt{-1}} \oint_{\mathcal{C}_{a,b,u,x}} \frac{x^s}{s^2} \frac{\zeta'(s)}{\zeta(s)}\, ds,$$

where $\mathcal{C}_{a,b,u,x} := ABGECDFHA$ is given as follows.

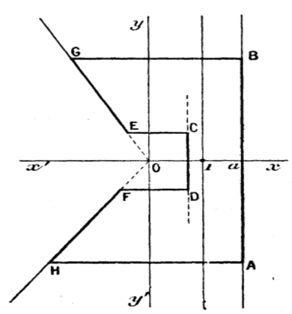

Figure 1: Contour integral

(a) (3 pts) Prove that there is a sequence $u_1 < u_2 < \cdots < u_k < \cdots$ of real numbers tending to infinity such that on the segments BG and AH (with $u = u_k$) our integral tends to zero. (Hint: Prove that $\left|\frac{\zeta'(s)}{\zeta(s)}\right| \le c \cdot u_k (\ln u_k)^2$ on BG and AH for $k = 1, 2, \cdots$.)

(b) (1 pt) Prove that on the segments GE and FH our integral tends to zero as $x, u \to +\infty$. (Hint: Prove that $\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq c \cdot |s|$ on GE and FH as $x, u \to +\infty$.)

(c) (2 pts) Prove that on the segments EC, CD, and DF our integral tends to zero as $x \to +\infty$. (Hint: Prove that for any $\delta > 0$, there is a constant $c > 0$ such that $|(s-1)\zeta(s)| \leq c \cdot e^{c|s|^{1+\delta}}$ for all $s \in \mathbb{C}$.)

(d) (1 pt) Prove that $\psi_2(x) \sim x$ as $x \to +\infty$. (Hint: Compute the residue.)